

659

N63-81910



JET PROPULSION LABORATORY  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
PASADENA, CALIFORNIA

RVT2854

763-81910

National Aeronautics and Space Administration  
Contract No. NASw-6


Technical Report No. 32-7

TRACKING AND ORBIT-DETERMINATION PROGRAM  
OF THE JET PROPULSION LABORATORY

Russell E. Carr  
R. Henry Hudson

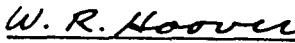
CARR, RUSS

HUDSON, R. H.

  
M. Eimer, Chief

Russell E. Carr  
Research Analysis Section

Copy No. \_\_\_\_\_

  
W. R. Hoover, Chief  
Applied Mathematics Section

JET PROPULSION LABORATORY  
California Institute of Technology  
Pasadena, California  
February 22, 1960

6668

---

Copyright © 1960  
Jet Propulsion Laboratory  
California Institute of Technology

---

JPL-TR-32-7

## CONTENTS

	Page
I. Introduction .....	1
II. Theoretical Basis for the Program .....	2
III. The System of Equations .....	8
IV. The Iterative Procedure .....	11
V. Calculated Values of the Observables and Partial's .....	13
VI. Auxiliary Output from the Program .....	22
VII. The IBM 704 Computer Program .....	26
VIII. Internal Program Control .....	30
IX. General Input .....	33
X. Miscellaneous Information .....	37
XI. Numerical Methods .....	43
Appendices	
A. Analytical Formulas for Partial's .....	44
B. Basis for Corrections for Refraction .....	51
C. Output Format .....	57

+

Page Intentionally Left Blank



## FIGURES

	Page
1. The Basic Computer Program .....	27
2. Procedure at Data Times .....	28
3. Generation and Solution of the Normal Equations .....	29
4. Transcription of Data Card to Data Tape .....	36
5. The $3\sigma^{ik}$ Rejection Test .....	41
B-1. Path of Electromagnetic Signal .....	56
B-2. Apparent Elevation Angle .....	56
C-1. Samples of Format .....	59

## ABSTRACT

The lunar-probe tracking program at the Jet Propulsion Laboratory has two prime objectives: (1) provide real-time predictions of the direction of the probe from various observation stations; (2) establish a reliable trajectory corresponding to the actual flight path of the probe. The tracking program, although developed for use with lunar probes, can be used for interplanetary probes if certain modifications are made.

The program, as developed for the IBM 704 digital computer, has two distinct phases. First, the equations of motion and the variational equations are integrated to each observation time where the elements of the equation  $\mathbf{A}\mathbf{u} = \mathbf{b}$  are computed. The second phase is concerned with the solution of a specified subset of  $\mathbf{A}\mathbf{u} = \mathbf{b}$ .

Flexibility and ease of operation have been major objectives in writing the 704 program. The number of data points and tracking stations that may be used is limited only by computing time and core storage. Input formats and operating instructions are presented for utilizing the various computational options available in the program.

## I. INTRODUCTION

The lunar-probe-tracking program of the Jet Propulsion Laboratory (JPL) has been developed using the maximum likelihood approach for correction of estimated values of the parameters defining the coasting trajectory. The correction of the estimate is obtained by correlating predicted values of certain quantities with the corresponding observed values.

The tracking program has two prime objectives: (1) provide real-time predictions of the direction of the probe from various observation stations in order that the antennas may acquire the signals being transmitted; (2) establish a reliable trajectory corresponding to the actual flight path of the probe.

The tracking program currently available at the IBM 704 digital-computer installation at JPL was developed for use with lunar probes. Although this tracking program can be used for interplanetary probes, certain modifications should be made to account for approximations which, though appropriate for flight paths of short duration, are unsuitable for interplanetary flights.

## II. THEORETICAL BASIS FOR THE PROGRAM

The tracking program is assumed to be effective during the coasting portion of the flight. The parameters chosen to define the calculated trajectory are the injection conditions

$$R_0, \phi_0, \theta_0, v_0, \gamma_0, \sigma_0 \quad (1)$$

at time  $t_0$ , where

$R \equiv$  distance from center of Earth, meters

$\phi \equiv$  geocentric latitude (positive north)

$\theta \equiv$  longitude from Greenwich (measured east)

$v \equiv$  magnitude of velocity relative to Earth, m/sec

$\gamma \equiv$  elevation angle of velocity relative to Earth (positive up from the local geocentric horizontal)

$\sigma \equiv$  azimuth angle of velocity relative to Earth (measured clockwise from north in local geocentric horizontal as viewed looking toward the center of the Earth)

at Greenwich Mean Time (GMT)  $t$ . (The time is actually measured in seconds from some reference time  $t_{\text{ref}}$ .)

It is assumed that the following quantities can be observed from the  $i$ th observation station:

$$r_i^{***}, f_i^{***}, \gamma_i^{***}, \sigma_i^{***}, \delta_i^{***}, \alpha_i^{***} \quad (2)$$



( )<sup>\*\*\*</sup>  $\equiv$  the effect of station aberrations have been included

$r_i \equiv$  distance from  $i$ th station to probe

$f_i \equiv$  doppler signal from probe at  $i$ th station

$\gamma_i \equiv$  local elevation angle of probe relative to  $i$ th station  
(geocentric horizontal)

$\sigma_i \equiv$  local azimuth angle of probe relative to  $i$ th station  
(geocentric horizontal)

$\delta_i \equiv$  local declination of probe relative to  $i$ th station

$\alpha_i \equiv$  local hour angle of probe relative to  $i$ th station

It is convenient to introduce the notation  $F^{ik}$  to designate the  $k$ th type of observation from the  $i$ th observation station:

$$F^{i1} \equiv r_i^{***}$$

$$F^{i2} \equiv f_i^{***}$$

$$F^{i3} \equiv \gamma_i^{***}$$

$$F^{i4} \equiv \sigma_i^{***}$$

$$F^{i5} \equiv \delta_i^{***}$$

$$F^{i6} \equiv \alpha_i^{***}$$

The quantity of the  $k$ th type which is actually observed from the  $i$ th observation station at time  $t_j$  will be denoted by  $\hat{F}_j^{ik}$ , which will differ from the true value of the observable  $F_j^{ik}$  by the observational error  $e_j^{ik}$ ; therefore,

$$\hat{F}_j^{ik} = F_j^{ik} + e_j^{ik} \quad (3)$$

It is assumed that the observational error consists of two parts,

$$e_j^{ik} = E^{ik} + \epsilon_j^{ik} \quad (4)$$

where  $E^{ik}$  is a constant, referred to as a bias, and  $\epsilon_j^{ik}$  is a random error from a normal distribution with mean zero and standard deviation  $\sigma_j^{ik}$ . Thus,

$$\hat{F}_j^{ik} = \tilde{F}_j^{ik} + \epsilon_j^{ik} \quad (5)$$

where

$$\tilde{F}_j^{ik} = F_j^{ik} + E^{ik} \quad (6)$$

Identity (6) might be written more completely as

$$\tilde{F}_j^{ik} (q_1, q_2, \dots, q_6, E^{ik}) = F_j^{ik} (q_1, q_2, \dots, q_6) + E^{ik} \quad (7)$$

where

$$q_1 \equiv R_0$$

$$q_2 \equiv \phi_0$$

$$q_3 \equiv \theta_0$$

$$q_4 \equiv v_0$$

$$q_5 \equiv \gamma_0$$

$$q_6 \equiv \sigma_0$$

From identity (7), it is evident that

$$\frac{\partial \tilde{F}_j^{ik}}{\partial q_l} = \frac{\partial F_j^{ik}}{\partial q_l} \quad (8)$$

for  $l = 1, 2, \dots, 6$ .

If  $q_l^{(r)}$  ( $l = 1, 2, \dots, 6$ ) represents the  $r$ th estimate of the injection conditions (1), and if  $E^{ikr}$  represents the  $r$ th estimate of the bias  $E^{ik}$ , then

$$\epsilon_j^{ik} = (\hat{F}_j^{ik} - \tilde{F}_j^{ikr}) - \sum_{l=1}^6 \frac{\partial F_j^{ikr}}{\partial q_l} \delta q_l^{(r)} - \delta E^{ikr} - \dots \quad (9)$$

where

$$\tilde{F}_j^{ikr} \equiv \tilde{F}_j^{ik} [q_1^{(r)}, q_2^{(r)}, \dots, q_6^{(r)}, E^{ikr}]$$

$$\frac{\partial F_j^{ikr}}{\partial q_l} \equiv \frac{\partial F_j^{ik}}{\partial q_l} \bigg|_{q_1 = q_1^{(r)}, \dots, q_6 = q_6^{(r)}}$$

$$\delta q_l^{(r)} \equiv q_l - q_l^{(r)}$$

$$\delta E^{ikr} \equiv E^{ik} - E^{ikr}$$

The assumptions made thus far on  $\epsilon_j^{ik}$  imply the frequency function

$$\frac{1}{\sqrt{2\pi} \sigma_j^{ik}} \exp \left[ -\frac{1}{2} \left( \frac{\hat{F}_j^{ik} - \tilde{F}_j^{ik}}{\sigma_j^{ik}} \right)^2 \right] \quad (10)$$

In setting up the likelihood function for the set of random observational errors  $\{\epsilon_j^{ik}\}$  associated with a set of observations  $\{\hat{F}_j^{ik}\}$ , two additional assumptions have been made:

1. The errors  $\epsilon_j^{ik}$  are independent with regard to time as well as to observational type and observation station.
2. The standard deviations  $\sigma_j^{ik}$  are not known, but for fixed  $i$  and  $k$ , the ratio of any two of the  $\sigma_j^{ik}$  is known; therefore

$$\sigma_j^{ik} = \frac{\sigma^{ik}}{w_j^{ik}} \quad (11)$$

where  $w_j^{ik}$  is known, may be written.

With these assumptions, the likelihood function associated with a set of observations  $\{\hat{F}_j^{ik}\}$  has the form

$$\prod_{i,k} \left\{ \left[ \left( \frac{1}{\sqrt{2\pi}} \right)^{N^{ik}} \right] \left( \prod_j \frac{w_j^{ik}}{\sigma^{ik}} \right) \right\} \exp \left\{ -\frac{1}{2} \sum_{i,k} \left[ \sum_j \left( \frac{w_j^{ik} \hat{F}_j^{ik} - w_j^{ik} \tilde{F}_j^{ik}}{\sigma^{ik}} \right)^2 \right] \right\} \quad (12)$$

where the products and the summations on  $j$  are understood to take place only on values of  $j$  corresponding to times  $t_j$  at which observations of the  $k$ th type have been made from the  $i$ th station. The number  $N^{ik}$  is the number of observations of the  $k$ th type from the  $i$ th observation station.

In addition, it is assumed that there is a frequency function, associated with the injection conditions and with the biases  $E^{ik}$ , of the form

$$\prod_{l=1}^6 \left( \frac{1}{\sqrt{2\pi} \sigma_{q_l}} \right) \prod_{i,k} \left( \frac{1}{\sqrt{2\pi} \sigma_{E^{ik}}} \right) \exp \left[ -\frac{1}{2} \sum_{l=1}^6 \left( \frac{q_l^{(r)} - q_l}{\sigma_{q_l}} \right)^2 - \frac{1}{2} \sum_{i,k} \left( \frac{E^{ikr} - E^{ik}}{\sigma_{E^{ik}}} \right)^2 \right] \quad (13)$$

where  $\sigma_{q_l}$  and  $\sigma_{E^{ik}}$  are *a priori* estimates, based on engineering knowledge, of the standard deviations of  $q_l$  and  $E^{ik}$  from their nominal values.

The resulting likelihood function  $L$  is merely the product of functions (12) and (13). The maximization of  $L$  with respect to the differences,  $\delta q_l^{(r)}$  and  $\delta E^{ikr}$ , leads to a set of nonlinear equations in these differences. The numerical solution of the nonlinear system is obtained by linearizing, replacing  $q_l - q_l^{(r)}$  and  $E^{ik} - E^{ikr}$  by  $q_l^{(r+1)} - q_l^{(r)}$  and  $E^{ik(r+1)} - E^{ikr}$ , respectively, and iterating until the linear system converges. The inclusion of the function (13) as a factor improves the conditioning of the linear system and, thus, accelerates convergence. The *a priori* estimates of the standard deviations enter in as reciprocal squares; therefore, setting the reciprocal squares to zero in the linear system corresponds to deletion of factor (13).

The tacit assumptions in any application using the procedure of iterating on the linear system to solve the nonlinear system are:

1. The initial estimate of the injection conditions and the biases must be sufficiently close to the maximum likelihood estimate of the injection conditions and biases.
2. Sufficient data to determine the solutions must be used.

### III. THE SYSTEM OF EQUATIONS

The system of equations resulting from linearization of the maximum likelihood equations may be written in the matrix form

$$Au = b \quad (14)$$

where the column matrix is partitioned

$$u = \begin{pmatrix} u_l \\ \vdots \\ u_s \end{pmatrix} \quad (15)$$

$$l = 1, 2, \dots, 6$$

$$s = 7, 8, 9, \dots$$

The sequence on  $s$  represents some well ordering of the biases  $E^{ik}$ . The elements of  $u$ , then, are

$$\begin{aligned} u_l &= q_l^{(r+1)} - q_l^{(r)} \\ u_s &= E^{ik(r+1)} - E^{ikr} \end{aligned} \quad (16)$$

The corresponding elements of the column matrix  $b$  are

$$b_l = \sum_{i,k} \left[ \frac{1}{(\sigma^{ikr})^2} \sum_j (w_j^{ik})^2 \frac{\partial F_j^{ikr}}{\partial q_l} (\hat{F}_j^{ik} - \tilde{F}_j^{ikr}) \right] \quad (17)$$

$$b_s = \frac{1}{(\sigma^{ikr})^2} \sum_j (w_j^{ik})^2 (\hat{F}_j^{ik} - \tilde{F}_j^{ikr})$$

and the elements of the symmetric matrix  $A$  are given by

$$A_{lm} = \sum_{ik} \left\{ \frac{1}{(\sigma^{ikr})^2} \sum_j \left[ (w_j^{ik})^2 \frac{\partial F_j^{ikr}}{\partial q_l} \frac{\partial F_j^{ikr}}{\partial q_m} \right] \right\} + \frac{\delta_{lm}}{(\sigma q_l)^2}$$

$$A_{lt} = \frac{1}{(\sigma^{ikr})^2} \sum_j (w_j^{ik})^2 \frac{\partial F_j^{ikr}}{\partial q_l} \quad (18)$$

$$A_{st} = \delta_{st} \left[ \frac{1}{(\sigma^{ikr})^2} \sum_j (w_j^{ik})^2 + \frac{1}{(\sigma_{E^{ik}})^2} \right]$$

where  $\delta_{pq}$  is Kronecker's delta, and

$$l, m = 1, 2, \dots, 6$$

$$s, t = 7, 8, 9, \dots$$

The quantity  $\sigma^{ikr}$  is the  $r$ th estimate of  $\sigma^{ik}$  and is given by

$$(\sigma^{ikr})^2 = \frac{1}{N^{ik}} \sum_j (w_j^{ik})^2 \left( \hat{F}_j^{ik} - \tilde{F}_j^{ikr} \right)^2 \quad (19)$$

In addition to  $\sigma^{ikr}$ , it is also desirable to have the quantity  $\sigma^{*ikr}$ , defined by

$$(\sigma^{*ikr})^2 = \frac{1}{N^{ik}} \sum_j (w_j^{ik})^2 \left[ \left( \hat{F}_j^{ik} - \tilde{F}_j^{ikr} \right) - \left( \sum_l \frac{\partial F_j^{ikr}}{\partial q_l} \Delta^r q_l + \Delta^r E^{ik} \right) \right]^2 \quad (20)$$

where

$$w = \left( \frac{\Delta^r q_l}{\Delta^r E^{ik}} \right) \quad (21)$$

is the solution to the matrix equation

$$Bw = c \quad (22)$$

with the elements of the  $7 \times 7$  symmetric matrix  $B$  given by

$$\begin{aligned}
 B_{lm} &= \sum_j (w_j^{ik})^2 \frac{\partial F_j^{ikr}}{\partial q_l} \frac{\partial F_j^{ikr}}{\partial q_m} \\
 B_{l7} &= \sum_j (w_j^{ik})^2 \frac{\partial F_j^{ikr}}{\partial q_l} \\
 B_{77} &= \sum_j (w_j^{ik})^2
 \end{aligned} \tag{23}$$

and the elements of  $c$  given by

$$\begin{aligned}
 c_l &= \sum_j (w_j^{ik})^2 \frac{\partial F_j^{ikr}}{\partial q_l} (\hat{F}_j^{ik} - \tilde{F}_j^{ikr}) \\
 c_7 &= \sum_j (w_j^{ik})^2 (\hat{F}_j^{ik} - \tilde{F}_j^{ikr})
 \end{aligned} \tag{24}$$

The quantity  $\sigma^{*ikr}$  serves as a measure of the noise on the data of the corresponding type.



#### IV. THE ITERATIVE PROCEDURE

At the beginning of the determination of the  $(r + 1)$ st estimate of the injection conditions  $q_l (l = 1, 2, \dots, 6)$  and the biases  $E^{ik}$ , it is necessary to have the set of observations  $\{\hat{F}_j^{ik}\}$  and the  $r$ th estimate of  $q_l$  and  $E^{ik}$ .

As corrections may not be made to the estimates of all injection conditions or estimates of biases, those which are to be corrected must be specified in order that the elements of column matrices (15) and (21) corresponding to those injection conditions and biases not being corrected may be set equal to zero, and the order of the matrix equations (Eqs. 14 and 22) may be correspondingly reduced. In the remainder of this Section of the Report, it is assumed that the reduced systems are being used when any reference is made to Eqs. (14) and (22). It should be added that, for simplification of the machine programming, biases are corrected for only one observation station at a time. Thus, the system of Eq. (14) is, at most, of 12th order.

The following steps indicate the iterative procedure, although they do not describe the sequence actually programmed for the computer.:

- (1) Calculate the set of  $\tilde{F}_j^{ikr}$
- (2) Calculate  $\sigma^{ikr}$  from Eq. (19)
- (3) Calculate the set of  $\frac{\partial F_j^{ikr}}{\partial q_l}$
- (4) Form the coefficients of  $Au = b$
- (5) Solve  $Au = b$  for  $q_l^{(r+1)} - q_l^{(r)}$  and  $E^{ik(r+1)} - E^{ikr}$ , where any difference not explicitly solved for is assumed to be zero.
- (6) Calculate

$$D_p^{(r)} = \sqrt{(\text{cofactor of } A_{pp}) / (\text{determinant of } A)} \quad (25)$$

for  $p = 1, 2, \dots$

- (7) Form the coefficients of  $Bw = c$
- (8) Solve  $Bw = c$  for  $\Delta^r q_l$  and  $\Delta^r E^{ik}$ , where any quantity not explicitly solved for is assumed to be zero
- (9) Calculate  $\sigma^{*ikr}$  from Eq. (20)

- (10) Replace  $q_l^{(r)}$  and  $E^{ikr}$  by  $q_l^{(r+1)}$  and  $E^{ik(r+1)}$ , respectively, and repeat the procedure.

One of the options available at the end of each iteration is that of predicting the value of one of the observables at some later time  $t_\mu$ . The prediction is accomplished by calculating

$$\tilde{F}_\mu^{ik(r+1)} \equiv \tilde{F}_\mu^{ik} [q_1^{(r+1)}, q_2^{(r+1)}, \dots, q_6^{(r+1)}, E^{ik(r+1)}, t_\mu] \quad (26)$$

as well as the quantity

$$\left\{ \left[ \frac{\partial F_\mu^{ik(r+1)}}{\partial q_1} D_1^{(r)} \right]^2 + \dots + \left[ \frac{\partial F_\mu^{ik(r+1)}}{\partial q_6} D_6^{(r)} \right]^2 \right\}^{\frac{1}{2}} \quad (27)$$

(referred to as the "standard deviation of the prediction"), where  $p$  indicates the particular member of (25) corresponding to the choice of  $i$  and  $k$ .

Another of the available options is that of putting the  $\hat{F}_j^{ik}$  observations to the test of a rejection procedure; the observation is rejected if

$$\left| \frac{w_j^{ik} [\hat{F}_j^{ik} - \tilde{F}_j^{ik(r+1)}]}{\bar{\sigma}^{ik}} \right| \geq K \quad (28)$$

where the denominator is inserted as an external constant for each data type, and  $K$  is a prescribed constant. Additional observations may be added to the original set of  $\hat{F}_j^{ik}$  in conjunction with this procedure.

The method of calculating  $\tilde{F}_j^{ikr}$  and  $\partial F_j^{ikr} / \partial q_l$  is discussed in the following Section.

## V. CALCULATED VALUES OF THE OBSERVABLES AND PARTIALS

The quantities  $\tilde{F}_j^{ikr}$  and the partial derivatives  $\partial \tilde{F}_j^{ikr} / \partial q_l$  are obtained numerically as a result of integrating a system of differential equations. This system may be divided into two parts: (1) an independent system which leads to the quantities  $\tilde{F}_j^{ikr}$ , and (2) a system dependent on the solution of the independent system and leading to the partial derivatives  $\partial \tilde{F}_j^{ikr} / \partial q_l$ . Both systems are integrated numerically relative to a rectangular coordinate system; transformations from the  $r$ th estimate of injection conditions (1) to the injection conditions

$$X_0^{(r)}, Y_0^{(r)}, Z_0^{(r)}, \dot{X}_0^{(r)}, \dot{Y}_0^{(r)}, \dot{Z}_0^{(r)} \quad (29)$$

are made by using the following equations at time  $t_0$ :

$$\begin{aligned} x &= R \cos \phi \cos \theta \\ y &= R \cos \phi \sin \theta \\ z &= R \sin \phi \\ \dot{x} &= v \sin \gamma \cos \phi \cos \theta - v \cos \gamma \sin \theta \sin \sigma - v \cos \gamma \sin \phi \cos \theta \cos \sigma \\ \dot{y} &= v \sin \gamma \cos \phi \sin \theta + v \cos \gamma \cos \theta \sin \sigma - v \cos \gamma \sin \phi \sin \theta \cos \sigma \\ \dot{z} &= v \sin \gamma \sin \phi + v \cos \gamma \cos \phi \cos \sigma \end{aligned} \quad (30)$$

$$\Theta = [\theta + \text{GHA } \Omega(t)] \bmod 2\pi \quad (31)$$

where GHA  $\Omega(t)$  is the Greenwich Hour Angle of the first point of Aries (i.e., vernal equinox) at time  $t$ ,

$$\begin{aligned} X &= R \cos \phi \cos \Theta \\ Y &= R \cos \phi \sin \Theta \\ Z &= z \end{aligned}$$

$$\begin{aligned}\dot{X} &= (\dot{x} - \omega y) \cos [\text{GHA } \Omega(t)] - (\dot{y} + \omega x) \sin [\text{GHA } \Omega(t)] \\ \dot{Y} &= (\dot{x} - \omega y) \sin [\text{GHA } \Omega(t)] + (\dot{y} + \omega x) \cos [\text{GHA } \Omega(t)] \\ \dot{Z} &= \dot{z}\end{aligned}\quad (32)$$

where  $\omega$  is the angular rate of the Earth's rotation.

It is convenient to introduce the notation

$$\begin{aligned}X_1 &\equiv X & X_1^{(0)} &\equiv X_0^{(r)} \\ X_2 &\equiv Y & X_2^{(0)} &\equiv Y_0^{(r)} \\ X_3 &\equiv Z & X_3^{(0)} &\equiv Z_0^{(r)} \\ X_4 &\equiv \dot{X} & X_4^{(0)} &\equiv \dot{X}_0^{(r)} \\ X_5 &\equiv \dot{Y} & X_5^{(0)} &\equiv \dot{Y}_0^{(r)} \\ X_6 &\equiv \dot{Z} & X_6^{(0)} &\equiv \dot{Z}_0^{(r)}\end{aligned}\quad (33)$$

The aforementioned independent system of differential equations may then be written

$$\frac{dX_p}{dt} = f_p(X_1, X_2, \dots, X_6, t) \quad (34)$$

with initial conditions

$$X_p = X_p^{(0)} \quad \text{at } t = t_0$$

for  $p = 1, 2, \dots, 6$ .

The dependent system of differential equations then may be written

$$\frac{d}{dt} \left[ \frac{\partial X_p}{\partial X_q^{(0)}} \right] = \sum_{m=1}^6 \frac{\partial f_p}{\partial X_m} \left[ \frac{\partial X_m}{\partial X_q^{(0)}} \right] \quad (35)$$

with initial conditions

$$\left[ \frac{\partial X_p}{\partial X_q^{(0)}} \right] = \delta_{pq} \quad \text{at } t = t_0$$

for  $p, q = 1, 2, \dots, 6$ , where  $\delta_{pq}$  is Kronecker's delta.

It is noted that the unknowns being solved for in Eq. (35) are the partial derivatives  $\left[ \partial X_p / \partial X_q^{(0)} \right]$ .

The functions  $f_p$  are given explicitly in the prior terminology by

$$\begin{aligned} f_1 &= \dot{X} \\ f_2 &= \dot{Y} \\ f_3 &= \dot{Z} \\ f_4 &= -\frac{K_E X}{R^3} g_1 - \sum_n K_n \left[ \frac{X - X^{(n)}}{r_n^3} + \frac{X^{(n)}}{R_n^3} \right] \\ f_5 &= -\frac{K_E Y}{R^3} g_1 - \sum_n K_n \left[ \frac{Y - Y^{(n)}}{r_n^3} + \frac{Y^{(n)}}{R_n^3} \right] \\ f_6 &= -\frac{K_E Z}{R^3} g_2 - \sum_n K_n \left[ \frac{Z - Z^{(n)}}{r_n^3} + \frac{Z^{(n)}}{R_n^3} \right] \end{aligned} \quad (36)$$

where

$$\begin{aligned} g_1 &= 1 + \frac{Ja^2}{R^2} \left( 1 - \frac{5Z^2}{R^2} \right) + \frac{Da^4}{7R^4} \left( \frac{63Z^4}{R^4} - \frac{42Z^2}{R^2} + 3 \right) \\ g_2 &= 1 + \frac{Ja^2}{R^2} \left( 3 - \frac{5Z^2}{R^2} \right) + \frac{Da^4}{7R^4} \left( \frac{63Z^4}{R^4} - \frac{70Z^2}{R^2} + 15 \right) \end{aligned} \quad (37)$$

where

$$r_n = \{ [X - X^{(n)}]^2 + [Y - Y^{(n)}]^2 + [Z - Z^{(n)}]^2 \}^{\frac{1}{2}}$$

$$R_n = \{ [X^{(n)}]^2 + [Y^{(n)}]^2 + [Z^{(n)}]^2 \}^{\frac{1}{2}}$$

$a$  is the Earth's equatorial radius,

$J$  and  $D$  are the Earth's coupling constants,

$K_E$  is the product of the universal gravitational constant and the mass of the Earth,  $K_E = GM_E$ ,

$K_n$  is the product of the universal gravitational constant and the mass of the  $n$ th body whose potential is being taken into account,  $K_n = GM_n$ , and  $X^{(n)}$ ,  $Y^{(n)}$ ,  $Z^{(n)}$  are the rectangular coordinates of the  $n$ th body at time  $t$ .

For the system of Eq. (35), it is assumed that the terms associated with  $J$  and  $D$  may be omitted, and that the explicit formulas used for the partial derivatives of the  $f_p$  are obtained by setting  $g_1 = g_2 = 1$  in Eq. (36) and performing the respective differentiations with respect to  $X$ ,  $Y$ ,  $Z$ ,  $\dot{X}$ ,  $\dot{Y}$ ,  $\dot{Z}$ . The following examples are typical of the twelve formulas which do not yield zero:

$$\frac{\partial f_1}{\partial \dot{X}} \equiv 1$$

$$\frac{\partial f_4}{\partial X} \equiv -K_E \left( \frac{1}{R^3} - \frac{3X^2}{R^5} \right) - \sum_n K_n \left\{ \frac{1}{r_n^3} - \frac{3[X - X^{(n)}]^2}{r_n^5} \right\}$$

$$\frac{\partial f_4}{\partial Y} \equiv \frac{\partial f_5}{\partial X} \equiv K_E \left( \frac{3XY}{R^5} \right) + \sum_n K_n \left\{ \frac{3[X - X^{(n)}][Y - Y^{(n)}]}{r_n^5} \right\}$$

(The remaining eight nonvanishing formulas are permutations of these four.)

The values used for  $\partial F_j^{ikr}/\partial q_l$  are obtained from

$$\frac{\partial F_j^{ikr}}{\partial q_l} = \sum_{m,p}^6 \frac{\partial F_j^{ikr}}{\partial X_p} \frac{\partial X_p}{\partial X_m^{(0)}} \frac{\partial X_m^{(0)}}{\partial q_l} \quad (38)$$

It is assumed that, so far as the partial derivatives  $\partial F_j^{ikr}/\partial q_l$  are concerned, one can use the calculated values of the partial derivatives of the functions  $F_j^{ik}$  which are not corrected for station aberrations or atmosphere. With this assumption, analytical formulas can be given for  $\partial F_j^{ikr}/\partial X_p$  and for  $\partial X_m^{(0)}/\partial q_l$ . These formulas are given explicitly in App. A. The middle factor in each term of the right side of Eq. (38) is given by the numerical solution of system (35) at time  $t_j$ .

The transformations from the numerical solution of system (34) at time  $t_j$  to  $\tilde{F}_j^{ikr}$  are made as follows:

The location of the  $i$ th observation station is given by

$$R_i, \phi_i, \theta_i \quad (39)$$

Then,

$$\begin{aligned} x_i &= R_i \cos \phi_i \cos \theta_i \\ y_i &= R_i \cos \phi_i \sin \theta_i \\ z_i &= R_i \sin \phi_i \end{aligned} \quad (40)$$

$$r_i = [(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2]^{\frac{1}{2}} \quad (41)$$

$$\hat{r}_i = \frac{(x - x_i) \hat{x} + (y - y_i) \hat{y} + (z - z_i) \hat{z}}{r_i} \quad (42)$$

$$\gamma_i = \arcsin \left[ \frac{(x - x_i) x_i + (y - y_i) y_i + (z - z_i) z_i}{R_i r_i} \right] \quad (43)$$

$$\dot{\gamma}_i = \frac{x_i \dot{x} + y_i \dot{y} + z_i \dot{z} - R_i \dot{r}_i \sin \gamma_i}{R_i r_i \cos \gamma_i} \quad (44)$$

The value of  $\sigma_i$  is given by

$$\begin{aligned}\sigma_i &= \sigma'_i \text{ if } [-(x - x_i) \sin \theta_i + (y - y_i) \cos \theta_i] \geq 0 \\ &= 2\pi - \sigma'_i \text{ if } [-(x - x_i) \sin \theta_i + (y - y_i) \cos \theta_i] < 0\end{aligned}\quad (45)$$

where, using principal values in the range  $0 \leq \sigma'_i \leq \pi$ ,

$$\sigma'_i = \arccos \left[ \frac{-(x - x_i) \sin \phi_i \cos \theta_i - (y - y_i) \sin \phi_i \sin \theta_i + (z - z_i) \cos \phi_i}{r_i \cos \gamma_i} \right] \quad (46)$$

Also,

$$\dot{\sigma}_i = \frac{\dot{x} \sin \phi_i \cos \theta_i + \dot{y} \sin \phi_i \sin \theta_i - \dot{z} \cos \phi_i + \cos \sigma_i (\dot{r}_i \cos \gamma_i - r_i \dot{\gamma}_i \sin \gamma_i)}{r_i \sin \sigma_i \cos \gamma_i} \quad (47)$$

$$\delta_i = \arcsin [(z - z_i)/r_i] \quad (48)$$

$$\dot{\delta}_i = (\dot{z} - \dot{r}_i \sin \delta_i)/r_i \cos \delta_i \quad (49)$$

$$\alpha_i = (\theta_i - \bar{\theta}_i) \bmod 2\pi \quad (50)$$

where, using values from the range  $0 \leq \bar{\theta}_i < 2\pi$

$$\bar{\theta}_i = \arctan [(y - y_i)/(x - x_i)] \quad (51)$$

(Ambiguities are resolved by observing the signs of numerator and denominator.)

$$\dot{\alpha}_i = \frac{(y - y_i) \dot{x} - (x - x_i) \dot{y}}{(x - x_i)^2 + (y - y_i)^2} \quad (52)$$



The values of  $\delta_i$  and  $\alpha_i$  may also be computed from the values of  $\gamma_i$  and  $\sigma_i$  by using the formulas

$$u_x = -\cos \gamma_i \sin \sigma_i \sin \theta_i - \cos \gamma_i \cos \sigma_i \sin \phi_i \cos \theta_i + \sin \gamma_i \cos \phi_i \cos \theta_i \quad (53)$$

$$u_y = \cos \gamma_i \sin \sigma_i \cos \theta_i - \cos \gamma_i \cos \sigma_i \sin \phi_i \sin \theta_i + \sin \gamma_i \cos \phi_i \sin \theta_i \quad (54)$$

$$u_z = \cos \gamma_i \cos \sigma_i \cos \phi_i + \sin \gamma_i \sin \phi_i \quad (55)$$

$$\delta_i = \arcsin u_z \quad (56)$$

$$\alpha_i = (\theta_i - \alpha'_i) \bmod 2\pi \quad (57)$$

where, using values from the range  $0 \leq \alpha'_i < 2\pi$

$$\alpha'_i = \arctan (u_y/u_x) \quad (58)$$

(Ambiguities are resolved by observing the signs of numerator and denominator.)

The angles  $\gamma_i$ ,  $\sigma_i$ ,  $\alpha_i$ ,  $\delta_i$  are subjected to a number of corrections:

1. The following correction is made for refraction:

$$\left. \begin{aligned} \gamma_i^* &= \gamma_i + \Delta \gamma_i (\gamma_i, R) & \text{if } 0 \leq \gamma_i \leq 15 \text{ deg} \\ &= \gamma_i + N_i \cot \gamma_i & \text{if } 15 \text{ deg} < \gamma_i \leq 90 \text{ deg} \end{aligned} \right\} \quad (59)$$

$$\sigma_i^* = \sigma_i$$

The basis for this correction is given in App. B. The increment  $\Delta \gamma_i$  is a function of  $\gamma_i$ , the elevation angle, and  $R$ , the probe's distance from the center of the Earth. The form suggested for  $\gamma_i > 15 \text{ deg}$  is a result of fitting the numerical results.

The corresponding values of  $\delta_i^*$  and  $\alpha_i^*$  are found by applying formulas (53) through (58) to  $\gamma_i^*$  and  $\sigma_i^*$ .

2. The following correction is made for the bubble vertical deviation from the radial vertical:

$$\left. \begin{aligned} \gamma_i^{**} &= \arcsin [\sin \gamma_i^* \sin \gamma_l + \cos \gamma_i^* \cos \gamma_l \cos (\sigma_l - \sigma_i^*)] \\ \sigma_i^{**} &= \arctan \left[ \frac{\sin \sigma_i^* \cos \gamma_i^* - \left( \frac{\pi}{2} - \gamma_l \right) \sin \gamma_i^* \sin \sigma_l}{\cos \sigma_i^* \cos \gamma_i^* - \left( \frac{\pi}{2} - \gamma_l \right) \sin \gamma_i^* \cos \sigma_l} \right] \end{aligned} \right\} \quad (60)$$

where the elevation angle and azimuth angle of the bubble vertical at the  $i$ th station (relative to the geocentric horizontal) are given by  $\gamma_l$  and  $\sigma_l$ .

The second equation of (60) is obtained by using the small angle approximations

$$\sin \gamma_l = 1$$

$$\cos \gamma_l = \frac{\pi}{2} - \gamma_l$$

$$\cos^2 \gamma_l = 0$$

in

$$\sigma_i^{**} = \arctan \left[ \frac{(\sin \sigma_i^* \sin \gamma_l \cos \gamma_i^* - \sin \gamma_i^* \sin \sigma_l \cos \gamma_l) \sin \gamma_l - \cos \gamma_i^* \cos \sigma_l \cos^2 \gamma_l \sin (\sigma_l - \sigma_i^*)}{\cos \sigma_i^* \cos \gamma_i^* \sin \gamma_l - \sin \gamma_i^* \cos \sigma_l \cos \gamma_l} \right]$$

(the exact formula). The ambiguity in the second equation of (60) is resolved by choosing that value of  $\sigma_i^*$  which differs the least from  $\sigma_l$ .

It is assumed that the bubble vertical deviation occurs only with observations made using a radio telescope having an azimuth-elevation angle mounting; therefore, no corresponding changes are made in  $\delta_i^*$  and  $\alpha_i^*$ .

Finally, bore-sight corrections are made:

$$\left. \begin{aligned} \gamma_i^{***} &= \gamma_i^{**} + \Delta \gamma_i \\ \sigma_i^{***} &= \sigma_i^{**} + \Delta \sigma_i \\ \delta_i^{***} &= \delta_i^* + \Delta \delta_i \\ \alpha_i^{***} &= \alpha_i^* + \Delta \alpha_i \end{aligned} \right\} \quad (61)$$

where  $\Delta \gamma_i$ ,  $\Delta \sigma_i$ ,  $\Delta \delta_i$ ,  $\Delta \alpha_i$  are simply additive constants which must be provided.

The quantity  $f_i^{***}$  is assumed to be a linear function of  $\dot{r}_i$ , the coefficients depending on the frequency transmitted as well as the manner in which the circuits are built for determining the doppler shift in frequency.

The formula presently being used for  $f_i^{***}$  has the form

$$f_i^{***} = \frac{f_{co} + B_i + C_i + \frac{f_{co}}{c} \dot{r}_i}{A_i} - F_i \quad (62)$$

where

$f_{co}$  is the transmitted frequency, cps.

$B_i$ ,  $C_i$ ,  $F_i$  are reference frequencies for the  $i$ th station, cps.

$A_i$  is a multiplicative factor in the circuitry used at the  $i$ th station.

$c$  is the velocity of light in a vacuum.

Currently, no correction in  $\dot{r}_i$  is made for the effects of refraction. The coefficient of  $\dot{r}_i$  in the linear function must also multiply the partial derivative of  $r_i^{***}$  in order to obtain the partial derivative of  $f_i^{***}$ . The range  $r_i$  is treated in a similiar manner.

It is assumed that the time derivatives of  $\gamma_i^{***}$ ,  $\sigma_i^{***}$ ,  $\delta_i^{***}$ , and  $\alpha_i^{***}$  are the same as those of  $\gamma_i$ ,  $\sigma_i$ ,  $\delta_i$ , and  $\alpha_i$ . The values of these time derivatives are useful in determining tracking rates required of the observing instruments.

## VI. AUXILIARY OUTPUT FROM THE PROGRAM

A number of quantities are calculated as auxiliary outputs of the program.

The transformations for obtaining

$$R, \phi, \theta, v, \gamma, \sigma \quad (63)$$

at time  $t$  from

$$X, Y, Z, \dot{X}, \dot{Y}, \dot{Z} \quad (64)$$

are given by

$$R = (X^2 + Y^2 + Z^2)^{\frac{1}{2}} \quad (65)$$

$$\phi = \arcsin \left( \frac{Z}{R} \right) \quad (66)$$

$$\theta = \arctan \left( \frac{Y}{X} \right) \quad (67)$$

(Ambiguities are resolved by observing the signs of numerator and denominator.)

$$\theta = [\theta - \text{GHA } \Omega(t)] \bmod 2\pi \quad (68)$$

$$\dot{x} = (\dot{X} + \omega Y) \cos [\text{GHA } \Omega(t)] + (\dot{Y} - \omega X) \sin [\text{GHA } \Omega(t)]$$

$$\dot{y} = -(\dot{X} + \omega Y) \sin [\text{GHA } \Omega(t)] + (\dot{Y} - \omega X) \cos [\text{GHA } \Omega(t)]$$

$$\dot{z} = \dot{Z} \quad (69)$$

$$v = [\dot{x}^2 + \dot{y}^2 + \dot{z}^2]^{\frac{1}{2}} \quad (70)$$

$$\gamma = \arcsin \left( \frac{x\dot{x} + y\dot{y} + z\dot{z}}{Rv} \right) \quad (71)$$

$$\begin{aligned} \sigma &= \sigma' \quad \text{if } (-\dot{x} \sin \theta + \dot{y} \cos \theta) \geq 0 \\ &= 2\pi - \sigma' \quad \text{if } (-\dot{x} \sin \theta + \dot{y} \cos \theta) < 0 \end{aligned} \quad (72)$$

where, using principal values in the range  $0 \leq \sigma' \leq \pi$

$$\sigma' = \arccos [(-\dot{x} \sin \phi \cos \theta - \dot{y} \sin \phi \sin \theta + \dot{z} \cos \phi)/v \cos \gamma] \quad (73)$$

A number of the auxiliary quantities calculated are related to the attitude of the probe. The equations of motion (Eq. 34) are those of the center of mass only, and their solution gives no information regarding the attitude. Although it is conceivable that dynamic equations may be added which would account for the calculation of the attitude of the probe at any time, the current assumption is that the attitude is known in terms of a unit vector  $\vec{c}$  in the direction of the probe axis.

The following quantities are also available as output from the program:

$$\angle \vec{c}, \vec{R}_{ip} = \arccos \left[ \frac{C_1(X - X_i) + C_2(Y - Y_i) + C_3(Z - Z_i)}{r_i} \right] \quad (74)$$

where  $C_1, C_2, C_3$  are the components of  $\vec{c}$  in the  $X, Y, Z$ , system and where  $\angle \vec{c}, \vec{R}_{ip}$  means the angle between  $\vec{c}$  and the vector from the  $i$ th station to the probe,

$$\angle \vec{c}, \vec{R}_{pn} = \arccos \left\{ \frac{C_1[X^{(n)} - X] + C_2[Y^{(n)} - Y] + C_3[Z^{(n)} - Z]}{r_n} \right\} \quad (75)$$

where  $\vec{R}_{pn}$  is the vector from the probe to the  $n$ th body whose potential is included,

$$\angle \vec{c}, \vec{R} = \arccos \left[ \frac{C_1X + C_2Y + C_3Z}{R} \right] \quad (76)$$

where  $\vec{R}$  is the vector from the center of the Earth to the probe,

$$V_n = |\vec{V}_n| = \{ [\dot{X}^{(n)}]^2 + [\dot{Y}^{(n)}]^2 + [\dot{Z}^{(n)}]^2 \}^{1/2} \quad (77)$$

where the time derivatives on the right are obtained from differencing ephemeris data for the  $n$ th body.

$$|\vec{V} - \vec{V}_n| = \{ [\dot{X} - \dot{X}^{(n)}]^2 + [\dot{Y} - \dot{Y}^{(n)}]^2 + [\dot{Z} - \dot{Z}^{(n)}]^2 \}^{1/2} \quad (78)$$

$$\angle(\vec{C}, \vec{V} - \vec{V}_n) = \arccos \left\{ \frac{C_1[\dot{X} - \dot{X}^{(n)}] + C_2[\dot{Y} - \dot{Y}^{(n)}] + C_3[\dot{Z} - \dot{Z}^{(n)}]}{|\vec{V} - \vec{V}_n|} \right\} \quad (79)$$

where  $\vec{V} - \vec{V}_n$  is the velocity of the probe relative to the  $n$ th body

$$\angle \vec{R}_{pn}, \vec{R}_{ns} = \arccos \left\{ \frac{[X^{(n)} - X][X_s - X^{(n)}] + [Y^{(n)} - Y][Y_s - Y^{(n)}] + [Z^{(n)} - Z][Z_s - Z^{(n)}]}{r_n |\vec{R}_{ns}|} \right\} \quad (80)$$

where  $\vec{R}_{pn}$  and  $\vec{R}_{ns}$  are, respectively, the vectors from the probe to the  $n$ th body and from the  $n$ th body to the Sun.

$$r_n = \frac{[X - X^{(n)}][\dot{X} - \dot{X}^{(n)}] + [Y - Y^{(n)}][\dot{Y} - \dot{Y}^{(n)}] + [Z - Z^{(n)}][\dot{Z} - \dot{Z}^{(n)}]}{r_n} \quad (81)$$

$$\angle \vec{R}_{En}, \vec{R}_{Ep} = \arccos \left[ \frac{X^{(n)} X + Y^{(n)} Y + Z^{(n)} Z}{R_n R} \right] \quad (82)$$

where  $\vec{R}_{En}$  and  $\vec{R}_{Ep}$  are the vectors from the center of the Earth to the  $n$ th body and the probe, respectively,

$$\angle \vec{R}_{ip}, \vec{R}_{is} = \arccos \left[ \frac{(X - X_i)(X_s - X_i) + (Y - Y_i)(Y_s - Y_i) + (Z - Z_i)(Z_s - Z_i)}{r_i |\vec{R}_{is}|} \right] \quad (83)$$

where  $\vec{R}_{ip}$  and  $\vec{R}_{is}$  are the vectors from the  $i$ th station to the probe and to the Sun, respectively, and

$$r_n = \{ [X - X^{(n)}]^2 + [Y - Y^{(n)}]^2 + [Z - Z^{(n)}]^2 \}^{\frac{1}{2}} \quad (84)$$

## VII. THE IBM 704 COMPUTER PROGRAM

As mechanized for the 704 digital computer, the tracking program can be considered as having two distinct phases. In the first phase, the equations of motion and the variational equations are integrated to each observation time where the elements of Eq. (14) are computed and stored. The second phase is concerned with the solution of a specified subset of Eq. (14). Figures 1 to 3 present block diagrams of the procedure.

The normal mode of operation is the iterative procedure where Eq. (14) is generated, solved, and the corrections are added to the prior estimate of the probe's injection coordinates. For increased flexibility, several alternatives are under operator control during this sequence of operations. The major options are as follows:

1. The composition of Eq. (14) may be specified as to (a) number of stations and data types to include (b) order of the matrix (c) weighting of the data types.
2. The data can be ignored if they fall  $3\sigma^{ik}$  or more outside the predicted value. (a) Data which fail the test are left out on subsequent iterations except that (b) all data may be re-examined at any time.
3. Pointing predictions for the tracking stations may be obtained at specified times using the latest values of the probe's injection coordinates. The "standard deviations of the predictions" may be computed.
4. During any iteration, the  $\hat{F}_j^{ik}$ ,  $\tilde{F}_j^{ik}$  and the partial derivatives  $\partial F_j^{ik}/\partial q_l$  may be printed off-line.



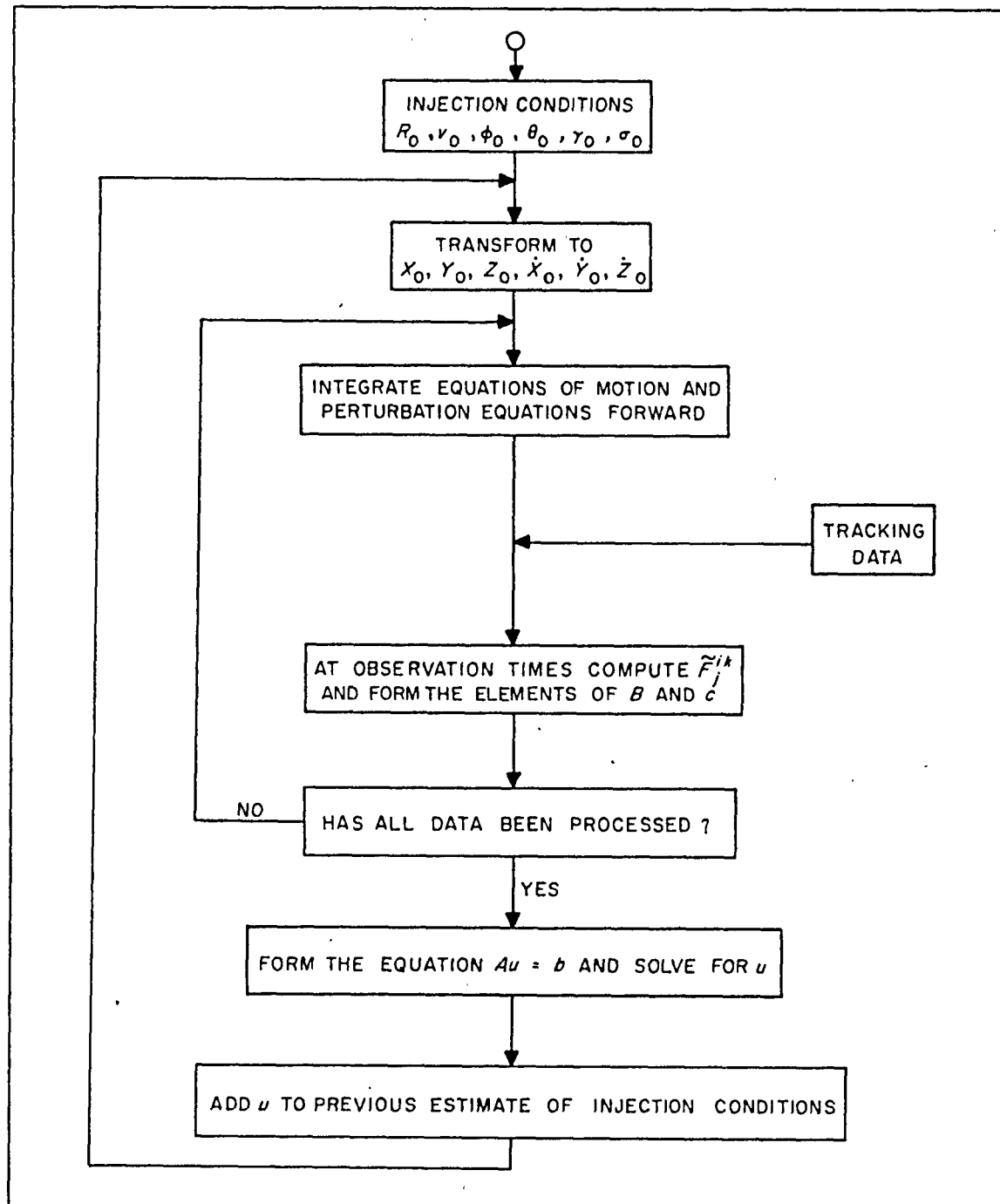


Fig. 1. The Basic Computer Program

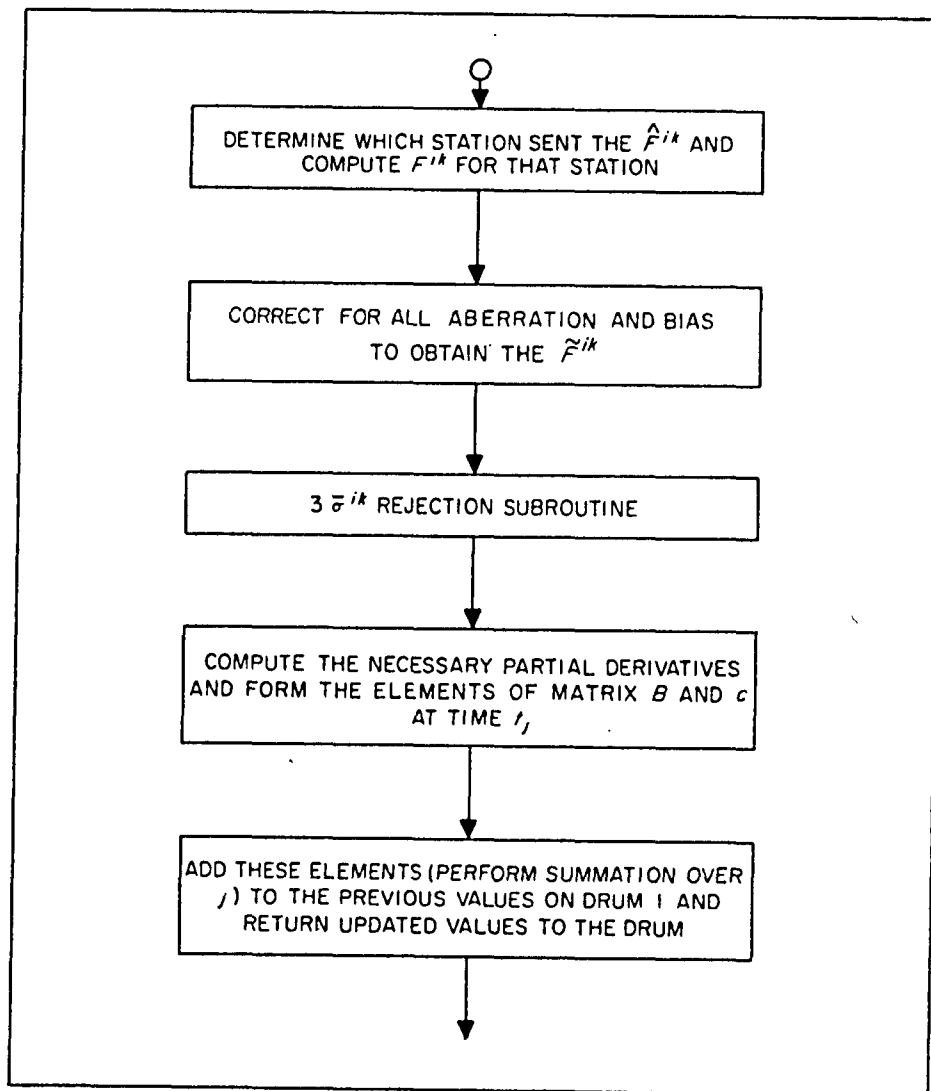


Fig. 2. Procedure at Data Times

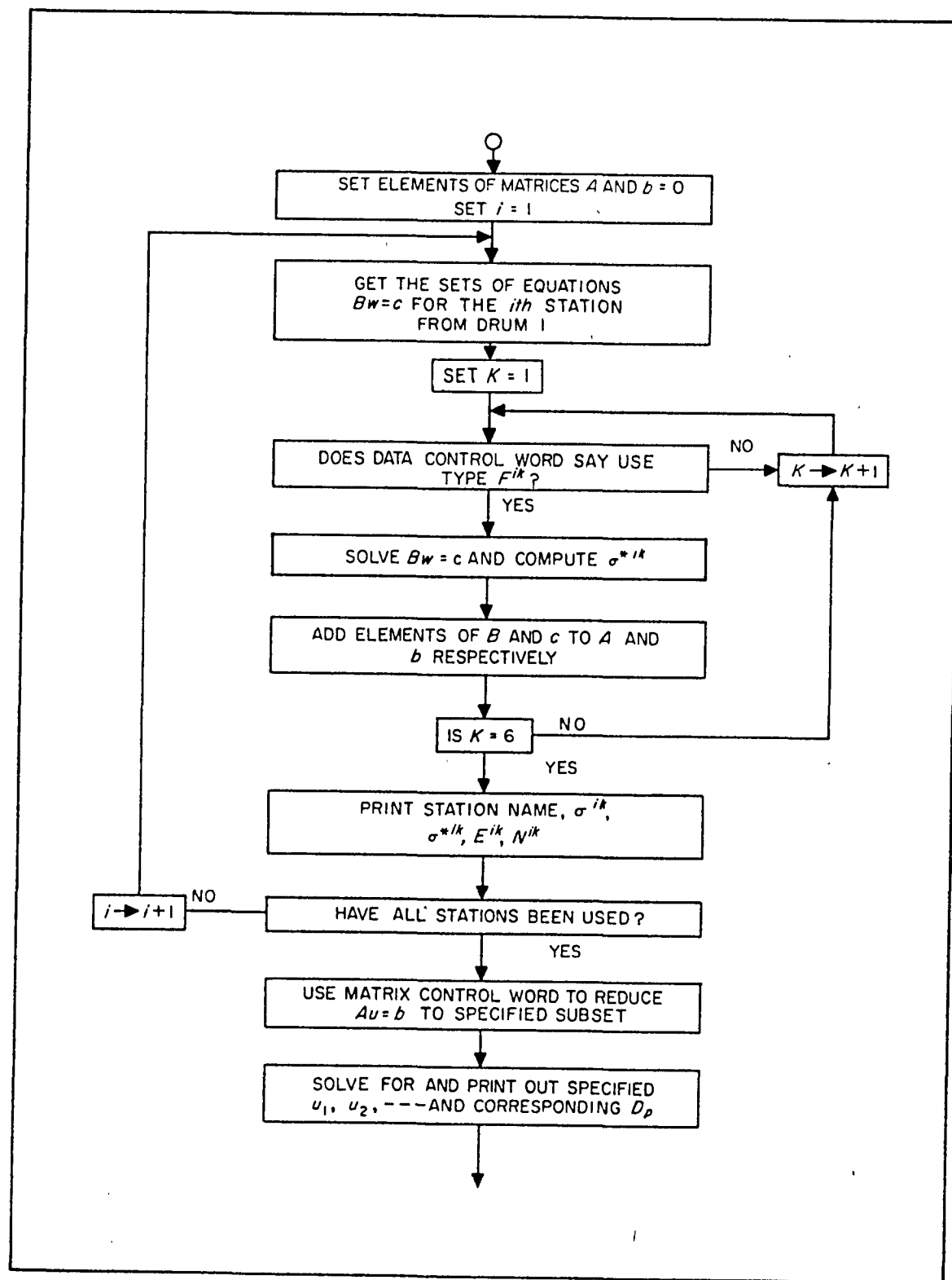


Fig. 3. Generation and Solution of the Normal Equations

## VIII. INTERNAL PROGRAM CONTROL

The input required by the program during any tracking operation may be divided into four groups:

- a. An estimate of the injection coordinates
- b. Data from tracking stations
- c. Internal constants
- d. Internal program control

Groups a, b, and c are defined by listing all the pertinent quantities in Sec. IX on input format.

The group d input is used primarily to specify the order and composition of the matrix equation (Eq. 14). In forming this equation, the program must ask the following questions:

- a. What data did the  $i$ th station send?
- b. Which data types are to be used?
- c. What subset of Eq. (14) is to be solved?

The control words to answer these questions are termed format control, data control, and matrix control respectively.

### A. Format Control

The first word on each tracking data card is assumed to be the station identification number  $i$ . The format control must identify the rest of the data on the cards. It should be noted that these control words are fixed in the sense that they are determined solely by the data sent from the tracking stations. The format control is an octal number associated with each tracking station of the form  $CCCCCCCC0000N$ . The  $C$  digits specify the type of data and the  $N$  specifies the amount. The first digit corresponds to the second word on the data card (remembering that the first word is  $i$ ), the second digit corresponds to the third word, etc. The amount,  $N$ , is the number of  $C$  digits used. The following tabulation gives the possible data types:

<i>C</i>	Data type
0	data condition
1	time ( $t_j$ )
2	slant range ( $\hat{F}^{i1}$ )
3	frequency ( $\hat{F}^{i2}$ )
4	elevation ( $\hat{F}^{i3}$ )
5	azimuth ( $\hat{F}^{i4}$ )
6	declination ( $\hat{F}^{i5}$ )
7	hour angle ( $\hat{F}^{i6}$ )

The data condition is a digit on the data card indicating the quality of the observations. The program accepts only data with a zero data condition. The minimum information that a card may contain is  $i$ ,  $t_j$ , and one observation. The maximum is  $i$  and the eight quantities above.

#### B. Data Control

The situation may occur in which the operator desires to omit a particular data type even though it appeared on the data cards. In order to accomplish this, another list of control words is kept by the program to specify the data types to be used in Eq. (14).

The data control words are octal numbers having the form  $ABABABABABAB$  where each pair  $AB$  represents a station. Six stations may be specified by one word: the left pair  $AB$  is associated with station 1, the right pair with station 6.

digit <i>A</i>	digit <i>B</i>
elevation    1	hour angle    1
frequency    2	declination    2
slant range    4	azimuth       4

In order to determine  $A$  and  $B$ , add the numbers corresponding to the data type desired. For example: in Eq. (14), use only slant range, frequency, hour angle, and declination from station 2. The data control word would be 006300000000.

### C. Matrix Control

Thus far, only the type of data has been specified. The order of Eq. (14) must also be entered as must the choice of weighting factors 1 or  $1/(c^{ik})^2$ . This is accomplished by one control word which is constructed as follows: If bit 2 equals 1, set  $c^{ik} = 1$ . If the program is to solve for a bias from the  $i$ th station,  $i$  must be entered in the decrement of the control word. The right-most 12 bits of the address are used to specify which quantities are to be solved for.

Bit number	Solve for correction to:
24	$R_0$
25	$\lambda_0$
26	$\phi_0$
27	$v_0$
28	$\omega$
29	$\dot{\omega}$
30	bias in $r_i$
31	bias in $f_i$
32	bias in $\dot{r}_i$
33	bias in $\dot{\phi}_i$
34	bias in $\dot{\lambda}_i$
35	bias in $\dot{v}_i$

The control word is checked for the two allowable cases,

1. decrement = 0, one or more of the bits 30-35 = 0
2. decrement = 0, bits 30-35 = 0

The program will stop on a divide check if it is asked to solve for a bias in an  $\hat{F}^{ik}$  which either did not appear in the tracking data or was omitted by means of the data control word.

### D. The Rejection Sigmas

A table of  $\sigma^{ik}$  is kept by the program for the purpose of rejecting bad data points. During any iteration, the observation  $\hat{F}_j^{ik}$  will not be used, by sense switch option, if  $|\hat{F}_j^{ik} - \tilde{F}_j^{ik}| \geq 3\sigma^{ik}$ . In order to conserve storage the  $\sigma^{ik}$  is packed in a pseudo-floating point form, three per word. As a station is allowed six types of observations, two words per station are required. The two words are decimal numbers of the form  $ExxExxExx$ ,  $ExxExxExx$ , where each triplet  $Exx$  represents a  $\sigma^{ik}$  and the order, left to right, is elevation, frequency, slant range, hour angle, declination, and azimuth. The decimal point is assumed at the right of  $xx$  and  $E$  is the exponent plus five: e.g., 815 15000., 515 15., 111 .0011. If the rejection option is being used and a triplet  $Exx = 0$ , it is interpreted as an instruction to keep that data type. Hence, rejection sigmas need be entered only for the data types which are suspected of containing bad points.

## IX. GENERAL INPUT

The input subroutine used by the program is *NYINP1*, which will accept data in the standard SAP form. Each card has a location field, an operation field, and a variable field. The location field, if needed, must contain the decimal address of the first word of data on the card. The operation field must contain *BCD*, *DEC*, *OCT*, or *TRA* specifying *BCD* information, decimal data, octal data, or a transfer. Floating point information is recognized by a decimal point or the letter *E* denoting multiplication by a power of 10; e.g.,  $1.88 \cdot 10^1 = 18.8E1$ . As with SAP, if data are to be stored sequentially in core, more than one word of data may be put on a card, each word being separated by a comma. A blank in the variable field indicates that information to the right of the blank is a comment and is not to be used.

## A. Format

The following tabulation gives the location and form of input data needed by the program. All *DEC* information is in floating point decimal form unless the identifying symbol is followed by an asterisk, in which event the information is a decimal integer. All storage locations are referred to by their symbolic locations, though decimal addresses must appear on the cards. A decimal symbol table is provided with the program listing.

The following dimensional units are used throughout the program:

length, meters  
velocity, m/sec  
angles, deg  
angular rates, deg/hr  
frequency, cps

columns	LOCATION FIELD 1-6	OPERATION FIELD 8-11	VARIABLE FIELD 12-72
GROUP 1	<i>R1</i>	<i>DEC</i>	<i>R<sub>0</sub></i>
	<i>R1</i> + 1	<i>DEC</i>	<i>v<sub>0</sub></i>
	<i>R1</i> + 2	<i>DEC</i>	<i>σ<sub>0</sub></i>
	<i>R1</i> + 3	<i>DEC</i>	<i>γ<sub>0</sub></i>
	<i>R1</i> + 4	<i>DEC</i>	<i>z<sub>0</sub></i>
	<i>R1</i> + 5	<i>DEC</i>	<i>σ<sub>0</sub></i>
	<i>R1</i> + 6	<i>DEC</i>	<i>σ<sub>c0</sub></i>
	<i>R1</i> + 7	<i>DEC</i>	<i>σ<sub>c0</sub></i>
	<i>R1</i> + 8	<i>DEC</i>	<i>t</i>

columns	LOCATION FIELD 1-6	OPERATION FIELD 8-11	VARIABLE FIELD 12-72
GROUP 2	(none)	DEC	$t^*, D^*, t^*, t_p, f_p, t_p, t_p, t_p, t_p$
GROUP 3	RI - i	DEC	$R_i$
	PHI - i	DEC	$\phi_i$
	THETA - i	DEC	$\theta_i$
	GI - i	DEC	$G_i$
	SI - i	DEC	$S_i$
	RBIAS - i	DEC	$\Delta r_i = E^{i1}$
	FBIAS - i	DEC	$\Delta f_i = E^{i2}$
	DIG - i	DEC	$\Delta \theta_i = E^{i3}$
	DIS - i	DEC	$\Delta \sigma_i = E^{i4}$
	DID - i	DEC	$\Delta \delta_i = E^{i5}$
	DIA - i	DEC	$\Delta \alpha_i = E^{i6}$
	ASUBI - i	DEC	$A_i$
	BSUBI - i	DEC	$B_i$
	CSUBI - i	DEC	$C_i$
	FSUBI - i	DEC	$F_i$
	HOLL1 - i	BCD	xxxxxx
	HOLL2 - i	BCD	xxxxxx
	FCOO	DEC	$f_{co}$
	VQS	DEC	$m^*$
GROUP 4	CODEWD - i	OCT	CCCCCCCCOOON
	DTYPE	OCT	ABABABABABAB
	REJSIG - 20 - 1)	DEC	ExxExxExx*, ExxExxExx* 5



In the preceding tabulation, the numbered input is not needed under the following assumptions:

1. Spin axis orientation is immaterial
2. No station misalignment (nominal values  $\gamma_I = 90^\circ$ ,  $\alpha_I = 0$  should be entered)
3. There are no constant biases to be subtracted from the data
4. The frequency computation is immaterial
5. The rejection option is not used

In group 1,  $t_0$  is a 10-digit decimal number giving GMT of injection. The word in pairs of digits is month, day, hour, minutes and seconds: e.g., 0113051648 is January 13, 5h, 16m, 48s. In group 2,  $D$  is the data condition and  $t$  is the GMT of the observation;  $t$  is a 6-digit number giving hours, minutes, and seconds. In order to specify the day of the year, the number of days since 0 hr January 1, 1959 must be punched in columns 70 through 72 on the first card of each day's data. As it reads the tracking data,  $NYINP1$  is modified in order that this number is not interpreted as a comment. It should be recalled that by means of the format control the amount of data following the  $i$  on each group 2 card is flexible and the data may be in any order. In group 3, the two BCD words are the first and last 6 letters of the station name, and  $m$  is the number of stations.

#### B. The Input Deck

All input from groups 1 and 4 (and group 3, if it was not written on the program tape) is stacked together and followed by the card  $TRA\ 3, 1$ . This card is followed by all the group 2 data.

Although the program sorts the group 2 data, with respect to time, in groups of 15 cards, the operator should take care that the data cards are not seriously out of time sequence. The group 2 cards are written on tape 2 as soon as they have been read. These cards are read only once, unless the operator decides to merge additional data with previously read cards. In this event, the operator will have to merge the cards and start over. Other than considerations of computing time, there is no limit on the number of group 2 cards that may be entered.

The group 2 input procedure is illustrated in Fig. 4.

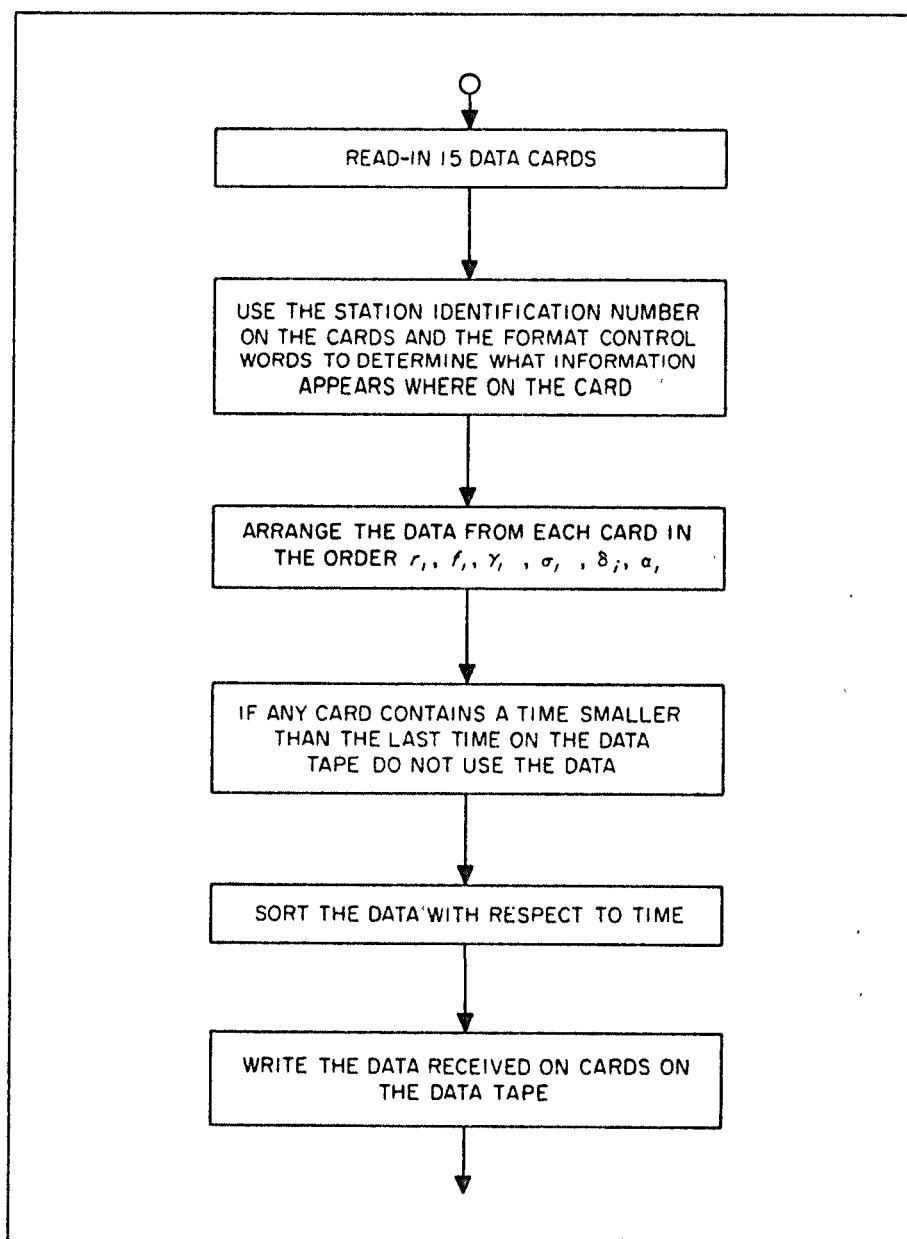


Fig. 4. Transcription of Data Card to Data Tape

## X. MISCELLANEOUS INFORMATION

### A. The Prediction Option

If sense switch 6 is down, the program will integrate past the last group 2 time and print the normal output at intervals specified by the operator. The prediction times may be specified in one of two ways:

$$PREDT DEC t', \Delta t, n \text{ or } PREDT DEC t'', \Delta t, n$$

where  $t'$ ,  $\Delta t$ ,  $n$  are floating point numbers, and  $t''$  is a decimal integer.

In the first case,  $t'$  is the number of seconds beyond the last group 2 card,  $\Delta t$  is the print interval in seconds, and  $n$  is the number of prediction points. In the second case,  $t''$  is the GMT of the first prediction with the same form as  $t_0$  in the group 1 cards;  $\Delta t$  and  $n$  mean the same as in the first case. Nominal values of 900., 1., 1. are currently assembled. This instructs the program to make one prediction 15 min after the last group 2 card. If sense switch 2 is down along with sense switch 6, Eq. (27) is evaluated and printed out at each prediction time.

### B. The Use of the Sense Switches

In the following tabulation, sense switch 2 is additionally tested as the program is loaded in core for on-line (down) or off-line output. Output is usually taken on-line as the operator must make decisions such as which data to use, when to reject bad points, and whether or not convergence has occurred.

Sense Switch	Up	Down
1	no operation	a. select card reader to read data while program is running b. if put down at LS1 + 2 stop, program returns to LS1 + 2 after solution of Eq. (14)
2	stop to enter matrix control word	a. if matrix control word $\neq 0$ , do not stop b. if SS#6 down, do standard deviations of the predictions.
3	no operation	print $\hat{F}_j^{ik}$ , $\tilde{F}_j^{ik}$ , and $\partial F_j^{ik} / \partial q_l$ on tape 4
4	no operation	do $3\sigma^{ik}$ rejection
5	all data on tape 2	more tracking data in card reader
6	no operation	predict

Sense switch 1 is tested at each integration step and merely selects the card reader, which is used to read in constants or control words which the operator may wish to change. Sense switches 3 and 4 are tested at each observation time. Sense switches 2, 5, and 6 are tested after the final observation is read from the data tape. Sense switch 5 should be up if there is no tracking data in the card reader.

### C. Program Halts and Error Indications

#### 1. Error stops (should not occur).

Location	Description
$INP + 60_8$	illegal operation in data card
$INP + 376_8$	out of range decimal data
$INP + 657_8$	illegal punch in data card
$BSTP - 5$	check sum error on tape 5 after 3 tries
$LOOKD - 10_8$	time out of range of ephemerides, check input
$LOOKD - 1$	time out of range of ephemerides, check input
$D2 + 7$	check sum error on tape 1, rewrite program tape
$START + 2$	$E\emptyset F$ return for card reader, input error if in tracking mode.

2. Error indications. If a 5-digit number appears on-line in columns 1 to 5, the octal location of an error return is indicated. These usually occur in the output transformations and do not affect the program: e.g., if  $\gamma = 90$  deg,  $\sigma$  is not defined and will give an error return. If there is a check sum error in reading the data tape, CHECK SUM ERROR ON DATA TAPE will be printed on-line.

### 3. Normal stops.

Location	
START - 3	Occurs after program deck or tape is loaded. If tracking, hit <i>Start</i> . If running single trajectory enter S, 1, 2 in <i>MQ</i> , hit <i>Start</i> .
LS 1 + 2	Enter matrix control word in <i>MQ</i> . If <i>SSW</i> 1 is down, program will return to this location to solve a different matrix. If the control word is zero or illegal, the program will remain at this stop.
TS 4	Program stops here after a prediction. If further iteration is desired, put <i>SSW</i> 6 up and hit <i>Start</i> .

#### D. Restart Feature

If the matrix *A* is ill-conditioned, the solution may be meaningless. The initial conditions from the previous iteration are saved; thus, if meaningless corrections are added to the injection conditions, they may be removed by hitting *Reset*, then *Start*. The program will return to the previous initial conditions and the operator can solve for a lower order matrix at the *LS* 1 + 2 stop.

#### E. Single Trajectories

It is possible to run the tracking program to obtain "single" trajectories. The variational equations are not integrated and computations pertinent only to tracking are ignored. The input consists of groups 1 and 3 followed by cards which specify the amount of printing to be done. These cards have the form

*TSTOP* + 6 DEC  $t_1, \Delta t_1, 1, t_2, \Delta t_2, 1, \dots, t_f, 0, 0$

*TRA* 3. 4

This instructs the program to begin printing at  $t_1$  with a print interval of  $\Delta t_1$  seconds. At  $t_2$ , switch to  $\Delta t_2$  seconds for the print interval, etc. As many as 18 time triplets may be entered. The triplet  $t_f, 0, 0$  instructs the program to select the card reader for another case at time  $t_f$ . The  $t$ 's are GMT and have the same format as  $t_0$  in group 1, whereas the  $\Delta t$ 's are entered as floating point numbers. As many cases as desired may be stacked together, separated by

TR 1 3, 4 cards. Only those parameters which change need be input on succeeding cases. After the program tape is loaded, enter bits S. 1, 2 in the *WQ* at the stop in location START - 3. Put the input deck in the card reader and hit *Start*. Output will be off-line on tape 3.

#### F. The Program Tape

Clear and load the binary deck. After the final card is read, the program will be written on tape 1. The operator may, at this time, write the group 3 data on the program tape. At the program halt in START - 3, put the group 3 data in the card reader, followed by the card *TRA DUMP*, and hit *Start*. The group 3 data will be written on the program tape. This may be done with any of the data but that of group 2. However, groups 1 and 4 are usually entered each time on cards as they are likely to change.

#### G. Atmospheric Refraction

In the present program a correction is added to the elevation angle to account for refraction effects. A two-parameter table look-up is made to determine this correction. As the current table in the program is for a transmitter frequency of 960 mc, the user may want to remove the atmospheric refractions correction. This may be done by putting the instruction *TRA STR* in location *SXD + 7*.

#### H. Rejection Logic

The program keeps a table of *ikj* binary bits corresponding to each observation  $\hat{F}_j^{ik}$ . If an observed value is rejected, the *ikj*th bit is set to 1 to indicate a bad data point. On subsequent iterations, instead of re-examining the data, this table is scanned to determine whether or not to use the  $\hat{F}_j^{ik}$  in the least-squares fit. Each time the sense switch option to reject bad points is used, the table is cleared and all data are re-examined for  $3\sigma^{ik}$  rejection (see Fig. 5).

#### I. Storage Allocation

The current JPL program has been written for a 704 computer with an 8K core, an 8K drum, and 5 tape units. Memory is distributed as follows: The program, tracking data, and Sun-Moon ephemerides are kept on separate tapes, leaving two units for off-line output. The *ik* normal matrixes and the *ikj* rejection bits are stored on the magnetic drums. All other data are kept in core memory. Although coded for *M* tracking stations, the current 8K core restricts the program to a maximum of five stations. A maximum on the order of 50 stations is anticipated for a 32K core with no magnetic drums.

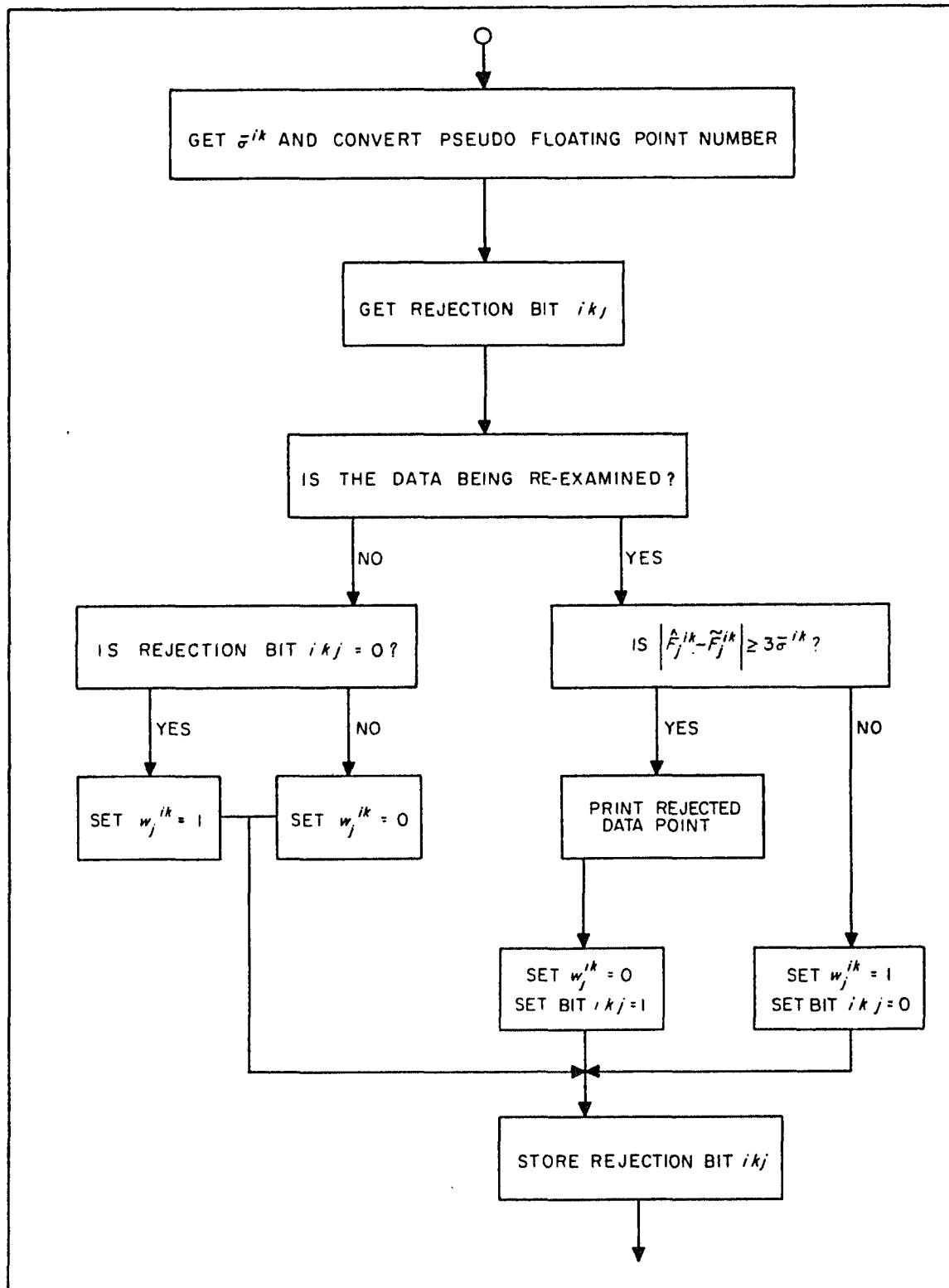


Fig. 5. The  $3\sigma^{ik}$  Rejection Test

Unit	Function
tape 1	program
tape 2	tracking data
tape 3	if SSW2; output
tape 4	if SSW3; $\hat{F}_j^{ik}$ , $\tilde{F}_j^{ik}$ and $\partial F_j^{ik} / \partial q_l$
tape 5	Sun-Moon ephemerides
drum 1	$ik$ normal matrixes
drum 4	$ikj$ rejection bits



## XI. NUMERICAL METHODS

A fourth-order Runge-Kutta method is used to integrate the equations of motion. This method was chosen primarily for the convenience of using a variable integration step which, as the tracking data may come in at any interval, is an important feature.

The matrix equations which appear in the least-squares method are often ill-conditioned, a situation which can cause serious loss of significant digits in the process of solving the equations. The program reduces this source of error as much as possible by using a Gauss-Jordan reduction method which interchanges rows and columns in order to obtain the largest pivotal element while developing the inverse matrix.

The current program uses a tape which contains the Cartesian coordinates of the Moon and Sun at 1/2-day intervals for the years 1959 through 1962. A fourth-order interpolating polynomial is used to find positions at any GMT. Efforts are currently under way to include an ephemeris of all the planets for the years 1960 through 1970.

As set up for tracking satellites and lunar probes, the program uses the following criteria for choosing integration step size:

$$\bar{R} = \min(R, R_n)$$

$0 \leq \bar{R} < 16000 \text{ km}$	$\Delta t = 30 \text{ sec}$
$16000 \leq \bar{R} < 86400$	360
$86400 \leq \bar{R}$	1800

These values have been determined empirically in order to ensure a small truncation error while, at the same time, giving relatively small computing times. A 35-hr trajectory to the Moon requires about 3 min computing time with a truncation error of less than 5 km at the Moon.

The computing time is severely increased when the program is in the tracking mode. The 36 variational equations must now be integrated and matrix computations must be made at each observation time. Further, the program is usually restricted from taking an optimum step size by the group 2 data rate. In the tracking mode, the computing time is on the order of 2 sec per observation time.

## APPENDIX A

## Analytical Formulas for Partial

$$\frac{\partial X_0}{\partial R_0} = \cos \phi_0 \cos \Theta_0 \quad (\text{A-1})$$

where

$$\Theta_0 = [\theta_0 + \text{GHA } \Omega(t_0)] \bmod 2\pi$$

$$\frac{\partial X_0}{\partial \phi_0} = -R_0 \sin \phi_0 \cos \Theta_0 \quad (\text{A-2})$$

$$\frac{\partial X_0}{\partial \theta_0} = -R_0 \cos \phi_0 \sin \Theta_0 \quad (\text{A-3})$$

$$\frac{\partial X_0}{\partial v_0} = 0 \quad (\text{A-4})$$

$$\frac{\partial X_0}{\partial \gamma_0} = 0 \quad (\text{A-5})$$

$$\frac{\partial X_0}{\partial \sigma_0} = 0 \quad (\text{A-6})$$

$$\frac{\partial Y_0}{\partial R_0} = \cos \phi_0 \sin \Theta_0 \quad (\text{A-7})$$

$$\frac{\partial Y_0}{\partial \phi_0} = -R_0 \sin \phi_0 \sin \Theta_0 \quad (\text{A-8})$$

$$\frac{\partial Y_0}{\partial v_0} = R_0 \cos \phi_0 \cos \Theta_0 \quad (\text{A-9})$$

$$\frac{\partial Y_0}{\partial \gamma_0} = 0 \quad (\text{A-10})$$

$$\frac{\partial Y_0}{\partial \gamma_0} = 0 \quad (A-11)$$

$$\frac{\partial Y_0}{\partial \sigma_0} = 0 \quad (A-12)$$

$$\frac{\partial Z_0}{\partial R_0} = \sin \phi \quad (A-13)$$

$$\frac{\partial Z_0}{\partial \phi_0} = R_0 \cos \phi_0 \quad (A-14)$$

$$\frac{\partial Z_0}{\partial r_0} = 0 \quad (A-15)$$

$$\frac{\partial Z_0}{\partial v_0} = 0 \quad (A-16)$$

$$\frac{\partial Z_0}{\partial \gamma_0} = 0 \quad (A-17)$$

$$\frac{\partial Z_0}{\partial \sigma_0} = 0 \quad (A-18)$$

$$\frac{\partial \dot{X}_0}{\partial R_0} = -\omega \cos \phi_0 \sin \Theta_0 \quad (A-19)$$

$$\frac{\partial \dot{X}_0}{\partial \phi_0} = -v_0 \cos \gamma_0 \cos \phi_0 \cos \sigma_0 \cos \Theta_0 - v_0 \sin \gamma_0 \sin \phi_0 \cos \Theta_0 + \omega R_0 \sin \phi_0 \sin \Theta_0 \quad (A-20)$$

$$\frac{\partial \dot{X}_0}{\partial \theta_0} = -v_0 \cos \gamma_0 \sin \sigma_0 \cos \Theta_0 + v_0 \cos \gamma_0 \sin \phi_0 \cos \sigma_0 \sin \Theta_0 - v_0 \sin \gamma_0 \cos \phi_0 \sin \Theta_0 - \omega R_0 \cos \phi_0 \cos \Theta_0 \quad (A-21)$$

$$\frac{\partial \dot{X}_0}{\partial r_0} = -\cos \gamma_0 \sin \sigma_0 \sin \Theta_0 - \cos \gamma_0 \sin \phi_0 \cos \sigma_0 \cos \Theta_0 + \sin \gamma_0 \cos \phi_0 \cos \Theta_0 \quad (A-22)$$

$$\frac{\partial \dot{X}_0}{\partial \gamma_0} = r_0 \sin \gamma_0 \sin \sigma_0 \sin \Theta_0 + r_0 \sin \gamma_0 \sin \phi_0 \cos \sigma_0 \cos \Theta_0 + r_0 \cos \gamma_0 \cos \phi_0 \cos \Theta_0 \quad (A-23)$$

$$\frac{\partial \dot{X}_0}{\partial \sigma_0} = -r_0 \cos \gamma_0 \cos \sigma_0 \sin \Theta_0 + r_0 \cos \gamma_0 \sin \phi_0 \sin \sigma_0 \cos \Theta_0 \quad (A-24)$$

$$\frac{\partial \dot{Y}_0}{\partial R_0} = \omega \cos \phi_0 \cos \Theta_0 \quad (A-25)$$

$$\frac{\partial \dot{Y}_0}{\partial \phi_0} = -r_0 \cos \gamma_0 \cos \phi_0 \cos \sigma_0 \sin \Theta_0 - r_0 \sin \gamma_0 \sin \phi_0 \sin \Theta_0 - \omega R_0 \sin \phi_0 \cos \Theta_0 \quad (A-26)$$

$$\begin{aligned} \frac{\partial \dot{Y}_0}{\partial \sigma_0} = & -r_0 \cos \gamma_0 \sin \sigma_0 \sin \Theta_0 - r_0 \cos \gamma_0 \sin \phi_0 \cos \sigma_0 \cos \Theta_0 \\ & + r_0 \sin \gamma_0 \cos \phi_0 \cos \Theta_0 - \omega R_0 \cos \phi_0 \sin \Theta_0 \end{aligned} \quad (A-27)$$

$$\frac{\partial \dot{Y}_0}{\partial r_0} = \cos \gamma_0 \sin \sigma_0 \cos \Theta_0 - \cos \gamma_0 \sin \phi_0 \cos \sigma_0 \sin \Theta_0 + \sin \gamma_0 \cos \phi_0 \sin \Theta_0 \quad (A-28)$$

$$\frac{\partial \dot{Y}_0}{\partial \gamma_0} = -r_0 \sin \gamma_0 \sin \sigma_0 \cos \Theta_0 + r_0 \sin \gamma_0 \sin \phi_0 \cos \sigma_0 \sin \Theta_0 + r_0 \cos \gamma_0 \cos \phi_0 \sin \Theta_0 \quad (A-29)$$

$$\frac{\partial \dot{Y}_0}{\partial \sigma_0} = r_0 \cos \gamma_0 \cos \sigma_0 \cos \Theta_0 + r_0 \cos \gamma_0 \sin \phi_0 \sin \sigma_0 \sin \Theta_0 \quad (A-30)$$

$$\frac{\partial \dot{Z}_0}{\partial R_0} = 0 \quad (A-31)$$

$$\frac{\partial \dot{Z}_0}{\partial \phi_0} = r_0 \sin \gamma_0 \cos \phi_0 - r_0 \cos \gamma_0 \sin \phi_0 \cos \sigma_0 \quad (A-32)$$

$$\frac{\partial \dot{Z}_0}{\partial \theta_0} = 0 \quad (\text{A-33})$$

$$\frac{\partial \dot{Z}_0}{\partial v_0} = \sin \gamma_0 \sin \phi_0 + \cos \gamma_0 \cos \phi_0 \cos \sigma_0 \quad (\text{A-34})$$

$$\frac{\partial \dot{Z}_0}{\partial \gamma_0} = v_0 \cos \gamma_0 \sin \phi_0 - v_0 \sin \gamma_0 \cos \phi_0 \cos \sigma_0 \quad (\text{A-35})$$

$$\frac{\partial \dot{Z}_0}{\partial \sigma_0} = -v_0 \cos \gamma_0 \cos \phi_0 \sin \sigma_0 \quad (\text{A-36})$$

$$\frac{\partial r_i}{\partial X} = \frac{X - R_i \cos \phi_i \cos \Theta_i}{r_i} \quad (\text{A-37})$$

where

$$\Theta_i = [\theta_i + \text{GHA } \Omega(t)] \bmod 2\pi$$

$$\frac{\partial r_i}{\partial Y} = \frac{Y - R_i \cos \phi_i \sin \Theta_i}{r_i} \quad (\text{A-38})$$

$$\frac{\partial r_i}{\partial Z} = \frac{Z - R_i \sin \phi_i}{r_i} \quad (\text{A-39})$$

$$\frac{\partial r_i}{\partial \dot{X}} = 0 \quad (\text{A-40})$$

$$\frac{\partial r_i}{\partial \dot{Y}} = 0 \quad (\text{A-41})$$

$$\frac{\partial r_i}{\partial \dot{Z}} = 0 \quad (\text{A-42})$$

$$\frac{\partial \dot{r}_i}{\partial X} = \frac{1}{r_i} (\dot{X} + \omega R_i \cos \phi_i \sin \Theta_i) - \frac{1}{r_i} \left( \dot{r}_i \frac{\partial r_i}{\partial X} \right) \quad (A-43)$$

$$\frac{\partial \dot{r}_i}{\partial Y} = \frac{1}{r_i} (\dot{Y} - \omega R_i \cos \phi_i \cos \Theta_i) - \frac{1}{r_i} \left( \dot{r}_i \frac{\partial r_i}{\partial Y} \right) \quad (A-44)$$

$$\frac{\partial \dot{r}_i}{\partial Z} = \frac{1}{r_i} \left( \dot{Z} - \dot{r}_i \frac{\partial r_i}{\partial Z} \right) \quad (A-45)$$

$$\frac{\partial \dot{r}_i}{\partial \dot{X}} = \frac{\partial r_i}{\partial X} \quad (A-46)$$

$$\frac{\partial \dot{r}_i}{\partial \dot{Y}} = \frac{\partial r_i}{\partial Y} \quad (A-47)$$

$$\frac{\partial \dot{r}_i}{\partial \dot{Z}} = \frac{\partial r_i}{\partial Z} \quad (A-48)$$

$$\frac{\partial \gamma_i}{\partial X} = \frac{1}{r_i \cos \gamma_i} \left( \cos \phi_i \cos \Theta_i - \sin \gamma_i \frac{\partial r_i}{\partial X} \right) \quad (A-49)$$

$$\frac{\partial \gamma_i}{\partial Y} = \frac{1}{r_i \cos \gamma_i} \left( \cos \phi_i \sin \Theta_i - \sin \gamma_i \frac{\partial r_i}{\partial Y} \right) \quad (A-50)$$

$$\frac{\partial \gamma_i}{\partial Z} = \frac{1}{r_i \cos \gamma_i} \left( \sin \phi_i - \sin \gamma_i \frac{\partial r_i}{\partial Z} \right) \quad (A-51)$$

$$\frac{\partial \gamma_i}{\partial \dot{X}} = 0 \quad (A-52)$$

$$\frac{\partial \gamma_i}{\partial \dot{Y}} = 0 \quad (A-53)$$

$$\frac{\partial \gamma_i}{\partial \dot{Z}} = 0 \quad (\text{A-54})$$

$$\frac{\partial \sigma_i}{\partial X} = \frac{1}{r_i \cos \gamma_i} (\sin \sigma_i \sin \phi_i \cos \Theta_i - \cos \sigma_i \sin \Theta_i) \quad (\text{A-55})$$

$$\frac{\partial \sigma_i}{\partial Y} = \frac{1}{r_i \cos \gamma_i} (\sin \sigma_i \sin \phi_i \sin \Theta_i + \cos \sigma_i \cos \Theta_i) \quad (\text{A-56})$$

$$\frac{\partial \sigma_i}{\partial Z} = \frac{1}{r_i \cos \gamma_i} (-\sin \sigma_i \cos \phi_i) \quad (\text{A-57})$$

$$\frac{\partial \sigma_i}{\partial \dot{X}} = 0 \quad (\text{A-58})$$

$$\frac{\partial \sigma_i}{\partial \dot{Y}} = 0 \quad (\text{A-59})$$

$$\frac{\partial \sigma_i}{\partial \dot{Z}} = 0 \quad (\text{A-60})$$

$$\frac{\partial \delta_i}{\partial X} = \frac{1}{r_i \cos \delta_i} \left[ \frac{\partial r_i}{\partial X} \frac{\partial r_i}{\partial Z} \right] \quad (\text{A-61})$$

$$\frac{\partial \delta_i}{\partial Y} = \frac{1}{r_i \cos \delta_i} \left[ \frac{\partial r_i}{\partial Y} \frac{\partial r_i}{\partial Z} \right] \quad (\text{A-62})$$

$$\frac{\partial \delta_i}{\partial Z} = \frac{1}{r_i \cos \delta_i} \left[ 1 - \left( \frac{\partial r_i}{\partial Z} \right)^2 \right] \quad (\text{A-63})$$

$$\frac{\partial \delta_i}{\partial \dot{X}} = 0 \quad (\text{A-64})$$

$$\frac{\partial \delta_i}{\partial \dot{Y}} = 0 \quad (\text{A-65})$$

$$\frac{\partial \delta_i}{\partial \dot{Z}} = 0 \quad (\text{A-66})$$

$$\frac{\partial \alpha_i}{\partial \dot{X}} = + \frac{1}{r_i \cos \delta_i} [\sin (\Theta_i - \alpha_i)] \quad (\text{A-67})$$

$$\frac{\partial \alpha_i}{\partial \dot{Y}} = - \frac{1}{r_i \cos \delta_i} [\cos (\Theta_i - \alpha_i)] \quad (\text{A-68})$$

$$\frac{\partial \alpha_i}{\partial \dot{Z}} = 0 \quad (\text{A-69})$$

$$\frac{\partial \alpha_i}{\partial \dot{X}} = 0 \quad (\text{A-70})$$

$$\frac{\partial \alpha_i}{\partial \dot{Y}} = 0 \quad (\text{A-71})$$

$$\frac{\partial \alpha_i}{\partial \dot{Z}} = 0 \quad (\text{A-72})$$



## APPENDIX B

## Basis for Corrections for Refraction

Consider an electromagnetic signal of given frequency sent from a probe to a receiving station fixed on the surface of the Earth. Assume an index of refraction

$$n = n(h) \quad (B-1)$$

where  $h = R - R_i$ . Suppose that the position of the probe (in the azimuth plane) at time  $t$ , relative to the observation station, is given by  $R(t)$ ,  $\gamma_i(t)$ , where  $R(t)$  is the distance to the probe from the center of the Earth and  $\gamma_i(t)$  is the elevation angle relative to the station (see Fig. B-1).

Because of an index of refraction which varies with increasing  $h$ , the path taken by the signal will not, in general, be a straight line but rather some curve such as  $C$ . The path  $C$  can be determined by using Fermat's principle and applying the calculus of variations.

Suppose one considers a variety of paths between the probe and the observation station. Fermat's principle states that the path taken will be that path  $C$  for which

$$\delta \int_C dt = 0 \quad (B-2)$$

Multiplying by the velocity of light in vacuum,  $c$ , replacing  $dt$  by  $ds/c'$  where  $c'$  is the velocity of light in the medium, and using the definition

$$n = c/c' \quad (B-3)$$

the relation (B-2) is replaced by

$$\delta \int_C n ds = 0 \quad (B-4)$$

Using the metric

$$ds^2 = dh^2 + (R_i + h)^2 (d\tau^*)^2 \quad (B-5)$$

where  $\tau^*$  is the central angle (see Fig. B-2), (B-5) becomes

$$\delta \int_C n \sqrt{1 + (R_i + h)^2 \left( \frac{d\tau^*}{dh} \right)^2} dh = 0 \quad (B-6)$$

Applying the calculus of variations to (B-6), the result is

$$\frac{d}{dh} \left\{ \frac{n(R_i + h)^2 \left( \frac{d\tau^*}{dh} \right)}{\left[ 1 + (R_i + h)^2 \left( \frac{d\tau^*}{dh} \right)^2 \right]^{\frac{1}{2}}} \right\} = 0 \quad (B-7)$$

Substituting

$$\gamma^*(h) = \arccot \left[ (R_i + h) \frac{d\tau^*}{dh} \right] \quad (B-8)$$

the result is

$$\frac{d}{dh} [n(R_i + h) \cos \gamma^*] = 0 \quad (B-9)$$

or

$$n(h) \cdot (R_i + h) \cdot \cos \gamma^*(h) = n(0) \cdot R_i \cdot \cos \gamma^*(0) \quad (B-10)$$

If the variable  $\xi^* = R_i \tau^*$  is introduced, Eq. (B-9) is replaced by

$$\frac{d\xi^*}{dh} = \frac{R_i}{R_i + h} \cot \gamma^*(h) \quad (B-11)$$

where

$$\cos \gamma^*(h) = \frac{n(0)}{n(h)} \frac{R_i}{R_i + h} \cos \gamma^*(0)$$

If there were no atmosphere,  $n(h)$  would always equal 1, and in this case, omitting the asterisk,

$$\frac{d\xi}{dh} = \frac{R_i}{R_i + h} \cot \gamma(h) \quad (B-12)$$

where

$$\cos \gamma(h) = \frac{R_i}{R_i + h} \cos \gamma(0)$$

Since at time  $t$ ,

$$\left. \begin{aligned} \gamma(0) &= \gamma_i(t) \\ \gamma^*(0) &= \gamma_i^*(t) \end{aligned} \right\} \quad (B-13)$$

the determination of the apparent elevation angle  $\gamma_i^*(t)$  from given  $R(t)$  and  $\gamma_i(t)$  consists of finding the initial condition  $\gamma^*(0)$  which is needed in order that the integration of (B-11) gives a value of

$$\xi_p^* = \int_0^{R(t)-R_i} \frac{R_i}{R_i + h} \cot \gamma^*(h) dh \quad (B-14)$$

equal to the value obtained by integration of (B-12),

$$\xi_p = \int_0^{R(t)-R_i} \frac{R_i}{R_i + h} \cot \gamma(h) dh \quad (B-15)$$

wherein the initial condition  $\gamma(0) = \gamma_i(t)$  is used.

In terms of the variables  $\xi^*$  and  $\xi$ , it is evident from Figs. A-1 and A-2 that the above determination is equivalent to requiring

$$\xi^* \Big|_{h=R(t)-R_i} = \xi \Big|_{h=R(t)-R_i} \quad (B-16)$$

a condition which must hold if  $C$  intersects the straight line path to the probe at the probe.

The numerical accomplishment of the determination can be done in the following manner:

A sequence of angles

$$\gamma^{*(1)}(0), \gamma^{*(2)}(0), \gamma^{*(3)}(0), \dots, \gamma^{*(k)}, \dots \quad (\text{B-17})$$

is built up by using

$$\left. \begin{aligned} \gamma^{*(1)}(0) &= \gamma_i(t), \text{ in degrees} \\ \gamma^{*(2)}(0) &= \left( 1.5 + \frac{88.5}{90} \gamma_i(t) \right), \text{ in degrees} \end{aligned} \right\} \quad (\text{B-18})$$

Integrals of type (B-14) are obtained with each of these initial conditions, giving, respectively,

$$\xi_p^{*(1)} \quad (\text{B-19})$$

and

$$\xi_p^{*(2)} \quad (\text{B-20})$$

Using the integral (B-15), the differences

$$\xi_p - \xi_p^{*(1)} \leq 0 \quad (\text{B-21})$$

and

$$\xi_p - \xi_p^{*(2)} \geq 0 \quad (\text{B-22})$$

are formed.

If  $\xi_p - \xi_p^{*(1)} = \xi_p - \xi_p^{*(2)} = 0$ ,  $\gamma_i^*(t) = \gamma_i(t) - 90$  deg. Otherwise, with  $k = 3$ , use

$$\gamma^{*(k)}(0) = \frac{|\xi_p - \xi_p^{*(2)}| \gamma^{*(1)}(0) + |\xi_p - \xi_p^{*(1)}| \gamma^{*(2)}(0)}{|\xi_p - \xi_p^{*(1)}| + |\xi_p - \xi_p^{*(2)}|} \quad (B-23)$$

Evaluate the integral of type (B-14) with initial condition  $\gamma^{*(k)}(0)$ , to obtain  $\xi^{*(k)}$ . If

$$|\xi_p - \xi_p^{*(k)}| > \epsilon \quad (B-24)$$

where  $\epsilon$  is some preassigned small non-negative number (in practice  $\epsilon$  is usually not chosen equal to zero), then, on the right side of (B-23), substitute the difference  $\xi_p - \xi_p^{*(k)}$  for the previous difference having the same sign and substitute  $\gamma^{*(k)}(0)$  for the corresponding angle. On the left side of (B-23), increase  $k$  by one, and repeat the process until a value of  $\xi_p^{*(k)}$  is obtained such that

$$|\xi_p - \xi_p^{*(k)}| \leq \epsilon \quad (B-25)$$

The corresponding value of  $\gamma^{*(k)}(0)$  is then used as  $\gamma_i^*(t)$ .

It is evident that the value of  $n(h)$  must be provided. For the current program, the assumption has been made that  $n(h)$  is of the form

$$n(h) = 1 + 10^{-6} G_i e^{-Ah} \quad (B-26)$$

where, if  $h$  is measured in statute miles,

$$A = 0.22$$

and  $G_i$  represents the surface refractivity for the  $i$ th observation station.

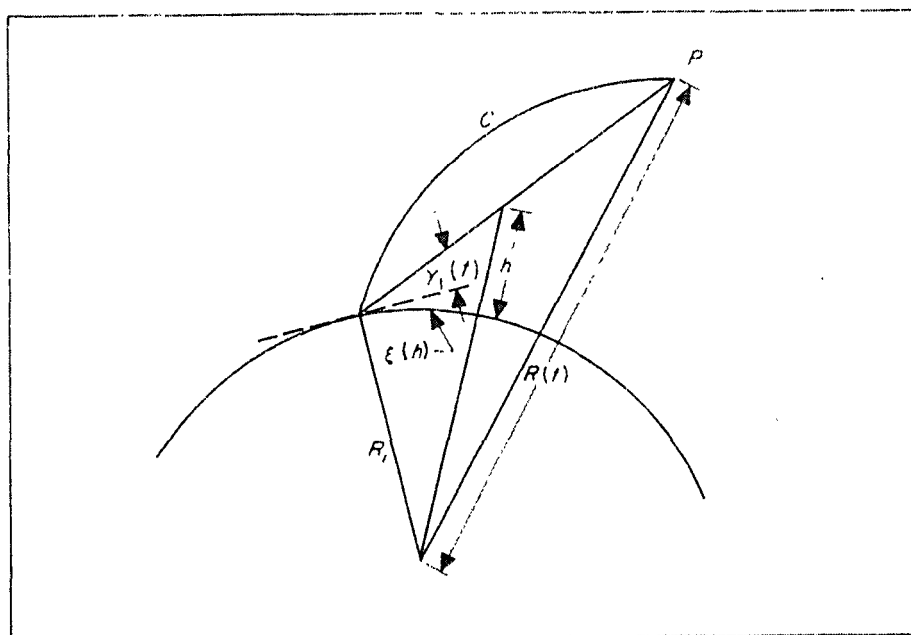


Fig. B-1. Path of Electromagnetic Signal

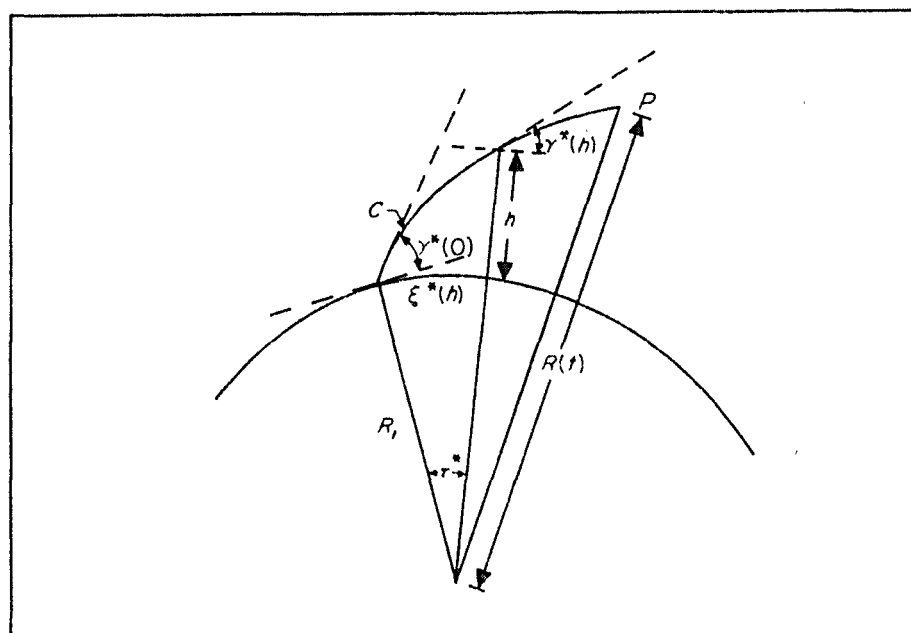


Fig. B-2. Apparent Elevation Angle

# APPENDIX C

## Output Format

The output is composed of three groups, the format of which are given in this Appendix. Group 1 is the normal trajectory output. Group 2 is the output obtained from the least-squares fit, and group 3 is the off-line output obtained from the sense-switch 3 option.

In group 1, the last line of output is obtained only if the option to compute the "standard deviations of the predictions" is used (see quantity 27). In group 2,  $A^{-1}$  is printed as a  $6 \times 6$  matrix if the rightmost 6 bits of the matrix control word are zero. Otherwise,  $A^{-1}$  is printed as a  $12 \times 12$  matrix with each two lines of printing representing one row of the matrix.

## GROUP 1

GMT		GMT- $t_0$		Year	
$R$	$v$	$\phi$	$\theta$	$\gamma$	$\sigma$
$\gamma_c$	$\sigma_c$	$\vec{x}_c, \vec{R}$	GHA $\Omega(t)$	$\Theta$	$V$
$X$	$\dot{X}$	$\ddot{X}$	$x$	$\dot{x}$	$C_1$
$Y$	$\dot{Y}$	$\ddot{Y}$	$y$	$\dot{y}$	$C_2$
$Z$	$\dot{Z}$	$\ddot{Z}$	$z$	$\dot{z}$	$C_3$

### Name of Nth Body

$X_n$	$\dot{X}_n$	$R_n$	$r_n$	$\vec{x}_c, \vec{R}_{pn}$	$\vec{x}_c, \vec{R}_{pn}, \vec{R}_{ns}$
$Y_n$	$\dot{Y}_n$	$\phi_n$	$\dot{r}_n$	$\vec{x}(c, \vec{V} - \vec{V}_n)$	$V_n$
$Z_n$	$\dot{Z}_n$	$\omega_n$	$ \vec{V} - \vec{V}_n $	$\vec{x} \vec{R}_{Fn}, \vec{R}_{Ep}$	0

### Name of ith Station

$\alpha_i$	$\delta_i$	$\sigma_i$	$\gamma_i$	$f_i$
$\dot{\alpha}_i$	$\dot{\delta}_i$	$\dot{\sigma}_i$	$\dot{\gamma}_i$	$\dot{r}_i$
$\vec{x}_c, \vec{R}_{ip}$	$\vec{x} \vec{R}_{ip}, \vec{R}_{is}$	$\phi_i$	$\theta_i$	$r_i$

### Name of ith Station

$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	$D_6$
-------	-------	-------	-------	-------	-------

# GROUP 2

## Normalization with Sigma

Name of <i>i</i> th Station	Sigma 1	Sigma 2	Bias	N
<i>R</i>	$\sigma^{i1}$	$\sigma^{*i1}$	$\Delta r_i$	$N^{i1}$
<i>F</i>	$\sigma^{i2}$	$\sigma^{*i2}$	$\Delta f_i$	$N^{i2}$
<i>EL</i>	$\sigma^{i3}$	$\sigma^{*i3}$	$\Delta \gamma_i$	$N^{i3}$
<i>AZ</i>	$\sigma^{i4}$	$\sigma^{*i4}$	$\Delta \sigma_i$	$N^{i4}$
<i>DEC</i>	$\sigma^{i5}$	$\sigma^{*i5}$	$\Delta \delta_i$	$N^{i5}$
<i>HA</i>	$\sigma^{i6}$	$\sigma^{*i6}$	$\Delta \alpha_i$	$N^{i6}$

## Noise Moment Matrix

$$A^{-1}$$

## Corrections to Initial Conditions

$u_1$	$u_4$	$u_2$	$u_3$	$u_5$	$u_6$
$u_7$	$u_8$	$u_9$	$u_{10}$	$u_{11}$	$u_{12}$
$D_1$	$D_4$	$D_2$	$D_3$	$D_5$	$D_6$
$D_7$	$D_8$	$D_9$	$D_{10}$	$D_{11}$	$D_{12}$

# GROUP 3

## GMT

<i>i</i>	$w^{ik}$	$\hat{F}^{ik}$	$\hat{F}^{ik} - \tilde{F}^{ik}$	$\tilde{F}^{ik}$	$\partial F^{ik} / \partial R_0$
	$\partial F^{ik} / \partial \phi_0$	$\partial F^{ik} / \partial \theta_0$	$\partial F^{ik} / \partial v_0$	$\partial F^{ik} / \partial \gamma_0$	$\partial F^{ik} / \partial \sigma_0$

For each GMT and each *i*, these three lines will be repeated *K* times, where *K* is the number of data types from the *i*th station. Samples of the output are presented in Fig. C-1.



03 MAR 05 15.25		00 00 00		1959	
7 659009	5 106401	2 281818	3 285774	2 120342	2 934874
-6 543389	3 360000	2 900000	3 239093	3 164867	5 110540
7-500743	4-499837	1 780841	7 157913	5 106086	455894
7 151645	4-984715	1-211167	7-559010	3 654199	-123290
7 311231	3 469722	1-434727	7 311231	3 469722	881453
MOON					
8-255926	4 103874	7 375142	9 377229	3 101052	3 104880
9-350025	3-107607	2-184081	4-936332	2 955477	4 104438
9-118463	2-124912	2 267040	5 114698	3 107990	000000
SUN					
12 140862	4 975523	12 148210	12 148216	2 689367	000000
11-422834	5 260537	1-710677	4 244244	3 107071	5 300271
11-183364	5 112990	5 104199	5 402912	3 158872	000000
HAWAII					
3 227999	1 777193	2 650915	2-357255	9 960020	
3-129698	2-209660	2-566568	3-121810	4 943169	
3 100264	3 128348	2 188235	3 204315	7 780168	
JODRELL BANK					
3 110121	3 341444	3 274454	2-267660	9 960065	
3-335187	2-337859	3-299684	3 174695	4-456593	
3 129223	3 135691	2 530510	3 357694	7 620343	
CANAVERAL					
3 284134	1 786064	2 896566	2 161304	5 104550	
3-470765	3-292280	2 478928	3-548468	5 104892	
2 924771	3 109543	2 282954	3 279420	6 669041	
GOLDSTONE					
3 256667	3 352092	2 886032	2-154199	5 105807	
3-158956	2 951701	3-173784	2-760349	4 999798	
3 110494	3 116107	2 352070	3 243152	7 408132	
PUERTO RICO					
3 111951	2 556770	3 328410	1 329563	5 122655	
3 151360	4 135059	4 123029	3 567065	4-561316	
2 371607	2 881727	2 180650	3 292911	7 137767	

Fig. C-1. Samples of Format. Group 1

NORMALIZATION WITH SIGMA					
PUERTO RICO	SIGMA1	SIGMA2	BIAS	N	
R	.0000	.0000	.0000		
F	.0000	.0000	.0000		
EL	2.1686	.0246	.0000	30	
AZ	.5713	.0211	.0000	30	
DEC	.0000	.0000	.0000		
HA	.0000	.0000	.0000		
GOLDSTONE	SIGMA1	SIGMA2	BIAS	N	
R	.0000	.0000	.0000		
F	.0000	.0000	.0000		
EL	.0000	.0000	.0000		
AZ	.0000	.0000	.0000		
DEC	4.2864	.0041	.0000	10	
HA	6.3258	.0055	.0000	10	
NOISE MOMENT MATRIX					
9 453242	4 146721	3-825884	6 696876	5-148040	3-922859
4 146721	-1 217607	-2-792041	1 804976	-1-653040	-1 308419
3-825884	-2-792041	-2 683842	1-597993	-1 454346	-2 168771
6 696876	1 804976	1-597993	4 687693	2-348524	1-215197
5-148040	-1-653040	-1 454346	2-348524	634947	-1 571974
3-922859	-1 308419	-2 168771	1-215197	-1 571974	145895
CORRECTIONS TO INITIAL CONDITIONS					
6 126241	3-121620	555782	-101311	1-431624	1 123789
000000	000000	000000	000000	000000	000000
5 212895	2 829272	147515	-1 826947	796836	381963
000000	000000	000000	000000	000000	000000

Fig. C-1 (Cont'd). Group 2

03 MAR 05 17 11					
1 1 10000	5 11435000	3 15050000	5 11264500	-3 74873738	
3-32386463	3 80175083	1 12017831	2 29025449	3-10357997	
03 MAR 05 17 11					
1 1 10000	2 19500000	1 12094400	2 16290560	-4 44734961	
1-25922729	-81205436	-2 15734382	63152147	42405450	
03 MAR 05 17 11					
1 1 10000	2 17508000	-84399700	2 18351997	-5-81731311	
1-15683623	1 52092538	-2 48278849	-44148571	21275023	
03 MAR 05 17 21					
1 1 10000	5 11355000	3 15025000	5 11204750	-3 40256253	
3-36546930	3 72706720	1 12620384	2 19458219	2-86128892	
03 MAR 05 17 21					
1 1 10000	2 20252000	84741807	2 19404582	-4 43198870	
1-25565827	1-10296101	-2 14982958	88608335	45780827	
03 MAR 05 17 21					
1 1 10000	2 22324000	-85962084	2 23183621	-5-84777583	
1-20039277	1 49194936	-2 49916649	-47395046	31075049	
03 MAR 05 17 31					
1 1 10000	5 11282000	3 14850000	5 11133500	-3 11683992	
3-39027717	3 65036476	1 12974494	2 10769599	2-70157784	
03 MAR 05 17 31					
1 1 10000	2 20816000	51687098	2 20299129	-4 41615510	
1-24781614	1-12205844	-2 14011162	93448831	48221737	
03 MAR 05 17 31					
1 1 10000	2 26908000	-82308936	2 27731089	-5-86320892	
1-23647999	1 45882686	-2 50647464	-49873461	40777286	
03 MAR 05 17 41					
1 1 10000	5 11217000	3 14575000	5 11071250	-3-10820597	
3-40096569	3 57591097	1 13135474	1 31410656	2-56113612	
03 MAR 05 17 41					
1 1 10000	2 21204000	21026969	2 20993730	-4 40030438	
1-23693790	1-13801022	-2 12944934	97697803	49789397	
03 MAR 05 17 41					
1 1 10000	2 31196000	-77243805	2 31968438	-5-86572025	
1-26501107	1 42358952	-2 50603247	-51623899	50020243	

Fig. C-1 (Cont'd). Group 3